APPROVAL PAGE [PLACEHOLDER]

DEVELOPMENT OF EMPIRICAL POSSIBILITY DISTRIBUTIONS IN RISK ANALYSIS

BY

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DISSERTATION

Submitted in Partial Fulfillment of the
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DEDICATION

This dissertation is dedicated to my mother, P. Shantha Bai for her unconditional love, support and encouragement in all aspects of my life. Her life-long commitment to education and her devotion to teaching will always be a source of motivation to me.

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ABSTRACT

The goal of this dissertation is to develop methods that can be used to systematically, consistently and logically generate uncertainty distributions that do not assume randomness in the system. Uncertainty is pervasive in all systems; even more so in systems such as risk assessment where information is accumulated from varied sources that are inherently complex and where the parameters of the system cannot be assumed to be random due to data insufficiencies. On the other hand, though non-probabilistic methods have been shown to be excellent tools to capture non-random uncertainty their application has been limited by their inability to offer methods for deriving distributions from empirical data. The motivation behind this dissertation is to find new models to overcome the difficulties inherent in the ability of probability theory to model non-random uncertainty, and to adapt these new models to capture both random and non-

random uncertainty as empirical possibility distributions. Two novel methods based on possibility theory are proposed to represent uncertainty in systems.

The property of consonance in data, where evidence leads an expert/model to inductively make judgments that converge to one possible outcome, is assumed in the development contained herein. Crisp notions of classical logic with a truth-value of either 0 or 1 (assuming complete evidence) on disjoint sets/outcomes ($A_i \cap A_j = \varnothing$) are replaced by softer notions where truth-values that range between 0 and 1 are defined over overlapping sets/outcomes ($A_i \cap A_i \neq \varnothing$). The new methods exploit set-based mechanisms such as interval analysis and cluster analysis to accomplish the task of deriving possibility distributions and to quantitatively represent imprecise knowledge. Relaxation of the axiom of additivity with the proper assumption of sub-additivity, allows the developments made here to represent inexact, imprecise, incomplete, and incoherent information in a more realistic and consistent manner. Two novel approaches for deriving possibility distributions are developed in this dissertation: i) Method I is used for non-consistent and non-disjoint data intervals; and ii) Method II is used for point estimates and disjoint data intervals. The new methods are illustrated through two case studies in human health risk assessment of radon gas exposure.

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Chapter 1 Introduction

In the modeling of any complex system one is almost always faced with the challenge of balancing costs with precision, usually sacrificing one for the other. In such circumstances a decision-maker has to contend with information that is incomplete, imprecise and uncertain, and is compelled to make decisions in a manner that is most appropriate and judicious. Lack of data and increasing costs of acquiring on-site and off-site data, constraints on time and the necessity to use information from varied sources (other experiments, expert judgments, and other structural information of the product being tested), pushes decision-makers to analyze associated uncertainties systematically and rigorously. Greater access to computational methods and the need for automating inference machines has driven the development of quantitative models that efficiently blend sparse empirical data with qualitative information. In this regard, quantitative methods have been developed in multiple areas of uncertainty analysis. In the probabilistic area, methods have been developed to include subjective information in uncertainty models thereby allowing an analyst to use expert judgments to fill in for missing data [Savage, 1976]. In non-probabilistic areas, theories such as evidence theory,

fuzzy set theory and possibility theory [Shafer, 1976; Zadeh, 1965; Dubois and Prade, 1988] have been developed to represent vague and ambiguous data. Technologies such as neural networks and genetic algorithms are being used more consistently to model complex artificial intelligence systems and systems based on non-monotonic reasoning are being pursued to develop knowledge-based systems [McDermott and Doyle, 1980].

1.1 Traditional First-Order Uncertainty Analysis

Traditionally, probability has been the predominant theory for modeling uncertainty in science and engineering, with the word uncertainty almost always synonymously associated with the word probability. A recent INSPEC search on "Title/Subject/Abstract" for the word "probability" for the period 1970-2003 yielded 170561 articles, as opposed to 34621 articles containing phrases such as fuzzy logic/fuzzy sets/possibility theory. Probability theory has been in existence for more than two centuries while non-probabilistic theories have been around for less than four decades, and hence this was expected; probability theory is a much more mature theory and technologists are more accustomed to its use. The debate is, however, not on the relevance or use of probability theory, but whether it is the right model to capture all forms of uncertainty. In classical probability theory one models uncertainties by assuming that the state of the world is defined by probability distributions depicting the randomness in systems, or, as Bayesians suggest, by depicting people's subjective beliefs in terms of the probability axioms. A long-standing interpretation of classical probability is that these distributions represent long run frequencies in a series of repeatable trials

[Von Misses, 1964]. Within the probability community, this definition has been challenged over the years as the need to build probability estimates from limited data and the need to include expert judgments became more urgent [Savage, 1976]. Inclusion of subjective expert data required modifying the definition of probability to transfer the state knowledge from the object to that of the observer, where the probabilities are more subjective and can be updated using Bayes' theorem of conditionalization. The essence of this theorem is that it allows one to update previously known or assumed probabilities (known as priors) as more information becomes available or as the state of the event for which the probability is being estimated changes with the realization of other events. This concept, that the probabilities can be updated with the availability of more data, lends the personalistic view of probability an advantage in that it enables one to estimate probabilities even in the absence of experimental data. This view, however, even if it is more amenable to modeling random behavior of the system from sparse data, still fails to capture all uncertainty. Due to its assumption of randomness and the dependence on the fundamental axiom of additivity, probability theory requires one to generate estimates that assume more information than what is available. Moreover, when very limited information is available, characteristics that are required by probability distributions, such as the shape and scale parameters, are rarely known with precision.

1.2 Second-Order Uncertainty Analysis

The recognition that point estimates insufficiently characterize uncertainty has resulted in using hierarchical models that attempt to alleviate the problem of point

estimation by specifying a second-order distribution on them. The form and nature of the models to derive and represent these second-order uncertainties has received much attention in the past few decades. These methods have ranged from those with strict probabilistic notions, such as Bayesian methods, probabilistic dependency bounds [Williamson and Down, 1990], to the application of such non-probabilistic theories as fuzzy set theory [Zadeh, 1965], evidence theory [Shafer, 1976] and more recently to the development of possibility theory [Dubois and Prade, 1988], theory of imprecise probabilities, and the theory of previsions [Walley, 1991; de Cooman, 2002]. The essence of these methods is that the first-order probabilistic estimates such as the mean or variance are no longer point estimates but are defined as a range of values. These values can be characterized as probabilistic distributions as used in Bayesian models or can be imprecise probabilities or vague estimates that are characterized by fuzzy sets.

1.3 Fuzzy Sets

Non-random uncertainty includes uncertainty arising out of vagueness in the description of the events, ambiguity in the identification of the occurrence of an event and the inability to completely characterize the occurrence of all the events in the universe of discourse. Inability of probability theory to model these other sources helped support the development of fuzzy set theory. Fuzzy set theory was initially developed to model subjective linguistic variables and has evolved over the past three decades into a mature method for uncertainty analysis. The essence of this theory is that it models vagueness in propositions. For example, a risk analyst could provide an estimate for the

magnitude of contamination as being "approximately 50 ppm" or determine the risk from being exposed to that approximate concentration to be "around 1E-04." Fuzzy set theory (FST) provides the calculus to quantify and use such a vague estimate in models. Though FST is useful in modeling the vagueness, it suffers from drawbacks that are similar to probability theory, in that it does not support all the sources of uncertainty, vis-à-vis randomness and ambiguity. There have been attempts, similar to the concept of second-order uncertainty analysis, to combine the two by modeling random behavior using probability theory and then using fuzzy logic and fuzzy arithmetic to model the vagueness in the probability estimates [Cooper and Ross, 1998]. The resulting uncertainty measures consist of an envelope of probability distributions at various levels of an analyst's “index of sureness” or a set of probability estimates resulting in a fuzzy membership function over the set of probabilities. This method of combining fuzziness and randomness, however, does not provide methods for the updating schemes and a consistent method to derive the distributions. An alternative to this approach is to use a model that provides inference mechanisms that can be used to directly elicit and represent uncertainties from incomplete and imprecise evidence.

1.4 Possibility Theory

Recently developed as an extension of evidence theory and fuzzy set theory, possibility theory provides a framework for a derivation of uncertainty when evidence is ambiguous and vague. It is especially useful when information is insufficient to warrant modeling randomness in the system. An interpretation of possibility theory can be

provided either by considering it a subset of Dempster-Shafer theory (also known as evidence theory), wherein belief functions are based on nested event structures, or a natural extension of fuzzy set theory wherein the membership functions define a possibility distribution. In evidence theory, Shafer extends the concept of lower and upper probabilities [Dempster, 1968] to model uncertainty measures as degrees of belief that are mathematically defined over sets of overlapping or non-overlapping events. In contrast to Dempster who views probabilities as aleatoric concepts, Shafer [1976] considers them as epistemic degrees of belief that are based on the subjects' belief or doubt in the occurrence of an event. This idea of epistemic belief allows evidence theory to model beliefs according to the reliability and accuracy of evidence supporting a particular proposition. This supporting evidence defines the weights or masses allocated to a set of propositions and then these weights can be combined to yield the combined belief for the corresponding set. For example, if the proposition was to determine the slope factor for a certain carcinogenic compound and some evidence showed that it was between 0.5 and 1.5, but other evidence also indicates that the value could be between 0.4 and 0.8, and still other evidence indicates it to be between 0.7 and 1.2, and so on, then, based on this information, an analyst can determine the two measures, belief and plausibility (defined in Chapter 3), for a proposition such as 'the value is between 0.6 and 1.2' using the calculus provided by evidence theory. These measures are analogous to the lower and upper bound probabilities. In contrast to evidence theory wherein these intervals can be partially overlapping, in possibility theory evidence is assumed to reveal information leading to only a consonant (nested intervals) set of propositions. In this theory, necessity and possibility measures are analogous to the belief and plausibility

measures, respectively. When considered as an extension of fuzzy set theory, possibility distributions are directly linked to fuzzy membership functions [Zadeh, 1978; Dubois and Prade, 1988]; however, it is not in this context that we use possibility in this dissertation.

One of the stated advantages of possibility theory is that it allows the analyst to acquire subjective as well as objective information in a form that is consistent with cognitive thought [Joslyn, 1994]. This is especially advantageous when the amount of information available is very limited and subjective judgments play an important role in estimation.

1.5 Risk Assessments and Uncertainty

. In a complex process such as risk assessment, it is extremely expensive, labor intensive, and time consuming to obtain information to support an exact simulation. In the absence of exact knowledge, an analyst is faced with the daunting task of choosing the most appropriate method for accounting for the uncertainties. Formal uncertainty analysis in risk assessments is still new, with most progress having occurred in the last decade or so. Uncertainties in health risk assessments were conventionally handled using conservative mechanisms such as default safety factors [NRC, 1994], 5th percentile probabilistic values and upper 95% confidence limits. These methods that yield point estimates, however, have been found to have significant financial costs and methodological shortcomings associated with them. For example, in non-carcinogenic risk assessment, the 10-fold safety factors applied to the reference doses lack firm biological basis and could in reality be higher or lower than a factor of 10 [Dourson and

Strata, 1983]. For exposure assessment analysis, it has been shown that the use of 95th percentile point estimates for various input variates (such as concentration of chemicals) can yield estimates of risk that are several orders of magnitude higher than those resulting from using entire distributions [Bogen, 1995; Burmaster and Harris, 1993]. In addition to these parametric uncertainties, there are a significant number of modeling uncertainties resulting from the inability to determine the right model for analyzing the potency of a chemical or determining the fate and transport of chemicals [Cornfield *et al*., 1980; NRC 1994].

In response to the increasing awareness of the drawbacks of the current methods, the context of models in risk assessment has drastically shifted from simplistic macro statistical models to molecular mechanistic models [Goddard and Krewski, 1995, Andersen, 1995; Hansch *et al*., 1995; Leung and Paustenbach, 1995]; and from conservative probability models yielding point estimates to more complete distributional models. On the mathematical front, scientists are attempting to improve the credibility of estimation by developing defensible numerical models that are more reflective of the true uncertainty; this includes the application of both the probabilistic methods [Burmaster, 1994, 1995; EPA, 1997] and also some non-probabilistic methods, most based on fuzzy set theory [Bardossy 1991, 1993; Ferson, 1992, 1993]. Given the complexity of risk assessments and scarcity of data, there is justification for more significant growth in the application of non-probabilistic methods such as possibility theory.

1.6 Contributions to the State-of-the-Art

One of the limitations of possibility theory is the lack of methods for deriving

possibility distributions from empirical data. Though possibility distributions can be elicited with relative ease for subjective judgments, methods for their empirical derivation have been very limited. Empirical measurements exist in almost all of science and engineering applications and the lack of methods for deriving possibility distributions restricts the use of possibility theory. Some significant contributions for developing possibility distributions from general measurements have been made in recent years [Joslyn, 1994, 1996], however these methods derive distributions that are sufficiently characterized by consistent measurements (where measurements are not required to be nested intervals, but it is only sufficient that all the measurements intersect). The intent of this dissertation is to extend these existing methods and to develop new methods for constructing set-based distributions that also satisfy the property of consonance (measurements reveal information focused on nested intervals). Useable possibility distributions are generated from non-consonant measurements by making them consonant for three types of measurements: 1) when measurements are overlapping interval data (not necessarily consistent but are not disjoint); 2) when measurements are singletons; and 3) when measurements are disjoint intervals.

Since the characteristics of overlapping interval data are fundamentally different from those of singletons or disjoint intervals, two separate methods are provided for each. Each measurement that is an overlapping interval corresponds to evidence that is focused on a collection of singletons, with each singleton being representative of a possible value and occurring in more than one interval. Thus, possibility distributions for overlapping interval measurements should be such that they infer the most possible value as being generated by the interval that is most persistent through all the measurements. Singletons

and disjoint sets, however, are disjoint and the possibility distributions infer the most possible value(s) as being the value(s) that have the highest possibility of occurring. The possibility structure for such measurements is derived such that each measurement is weighted according to its proximity to the most possible value. Thus, possibility distributions for overlapping measurements are derived by considering intersections of all measurements while those based on disjoint or singleton measurements are derived using clustering techniques (see Chapter 3). Possibility distributions thus derived are considered to be relevant substitutes to probability distributions or complementary to probability distributions. The dissertation also presents an extensive review of uncertainty analysis in risk assessments and two case studies to illustrate the use of possibility theory to risk assessments.

1.7 Outline

The intent of the research leading up to this dissertation was to determine the state of art of uncertainty analysis in human health risk assessments and to explore the applicability of new uncertainty methods to risk models. As part of this exploration process a thorough review of uncertainty in risk assessments was conducted. The review examined the sources of uncertainty in all areas of risk assessment, current methods for handling these uncertainties, and their limitations. In addition, two novel theoretical developments came out of this exploration, and these two are described in Chapter 3 (a method for constructing possibility distributions from non-consonant and non-consistent random intervals, and the construction of possibility distributions from point estimates).

Chapter 2 presents the results of the examination of sources of uncertainty in risk assessment. Risk assessment is a complex process consisting of four main areas: i) hazard identification; ii) exposure analysis; iii) toxicity assessment; and iv) risk characterization. While all the areas of risk assessment could encompass all the types of uncertainty, the magnitude of each type is specific to the area. For example, in hazard identification, subjectivity and vagueness might be more prevalent, while in exposure characterization, randomness might be more dominant. An analysis of the review resulted in identification of a number of areas in which non-probabilistic methods can be beneficial in aiding an analyst at an exploratory stage or at an advanced stage of risk analysis. A sample of the areas in which these methods are already being used is presented in Section 2.2.2. Currently, the non-probabilistic methods have predominantly been those that are based on fuzzy set theory. Though fuzzy set theory offers an effective model for representing uncertainties in risk assessments, it is most useful for representing uncertainties that are naturally expressed as linguistic variables rather than for representing uncertainty that results from not understanding an inferential mechanism.

The purpose of Chapter 3 is to present possibility theory as a viable model for representing uncertainties that can be directly developed from empirical data. The main emphasis of this chapter is to articulate the two theoretical developments discovered in the course of this research project, which both provide methods to systematically develop possibility distributions from measurements represented as overlapping intervals or from measurements that are disjoint such as point estimates or non-overlapping intervals. Development of these novel methods is essential in laying the groundwork for the application of possibility theory to risk analysis. These methods enable one to develop

possibility distributions that can subsequently be used either as a substitute (first-order representations) for, or complementary (second-order representations) to, probability distributions.

Chapter 4 presents a case study to illustrate the main points of the novel methods developed herein. How an uncertainty analysis is conducted is in many ways dependent on what is being analyzed and what the impact of the analysis is. In the various areas of risk assessments, uncertainty is sometimes reduced through rigorous applications of physical and biological models while in some cases the complex models are simplified. Due to its independence from the requirements of randomness and dependence on weaker axioms, possibility theory offers an adaptable mechanism for representing uncertainties at any level of risk assessment. In Chapter 4, development of possibility distributions is illustrated through a previously conducted meta-analysis, which was based on a standard statistical approach. There are a number of creative ways in which possibility theory can be used in risk assessments and the material in this dissertation addresses a small, but important, number of these.

Finally, Chapter 5 presents the conclusions of this dissertation and identifies areas of future research that will advance the application of possibility theory to risk assessments and other scientific applications.

Chapter 2 Uncertainty in Risk Assessment

This Chapter presents the definitions of various uncertainty formalisms and the current state of the art of uncertainty analysis in risk assessments.

Uncertainty in the real world is of many types. Many in the scientific community have construed these various types of uncertainty and its definition in various ways and have effectively distinguished one type from another to better aid in the modeling of uncertainty [Shafer and Pearl, 1990; Parsons and Hunter, 1998]. In general, it is understood that the many types of uncertainty arise from varied sources and are handled according to their origin. To better model these various types of uncertainty, analysts usually segment uncertainty into three types: 1) uncertainty due to randomness - exists when evidence points to the occurrence of conflicting events, that is any of the events can occur with a given probability; 2) uncertainty due to vagueness - occurs due to the inability to define distinct boundaries for any set for which the uncertainty measure is being calculated; and 3) uncertainty due to ambiguity - the result of the indecision in assignment of an element to a given set, i.e., one is sure about the occurrence of an event but unsure about which one. Amongst these various types, uncertainty due to randomness has been given the most attention, and historically has been studied

extensively using probability theory. In contrast, uncertainty due to vagueness, usually represented by fuzzy sets, has been addressed in the literature in the recent past as compared to probability theory, and continues to distinguish itself as a separate paradigm from probability theory in the uncertainty it models; its main applications being in areas where vagueness plays a significant role - as in when modeling linguistic variables. Finally, ambiguity, which is a more subtle type of uncertainty wherein evidence is insufficient to assign the occurrence to any crisp event, has usually been handled using rough sets [Pawlak, 1982] and possibility theory. In most applications all these forms of uncertainty can co-exist, hence, a more thorough model to blend these has resulted in research to conceptually fit the existing models to cases where all the forms exist. In later sections two such models, evidence theory and possibility theory, are presented.

Before attempting to introduce the specifics of each of the uncertainty modeling theories, a brief summary of the uncertainty sources pertaining to risk assessment is presented. A more thorough understanding of the sources of uncertainty helps demarcate the various representative types and aids the analyst in choosing the right tool to use.

2.1 Sources of Uncertainty

Within the risk assessment community it is broadly accepted that uncertainty can be a result of randomness, bias, or variability. Randomness is distinguished from variability in that it arises from the imperfection in knowledge about the true value of the variable being measured, while variability, on the other hand, is the real variation of the property being measured among individuals [Hattis and Burmaster, 1994; Hattis and Silver, 1994]. For example, the heights of various individuals within a general

population can truly vary according to the gender, geographical location and age, while the heights of individuals picked randomly from the population within a specific category, such as men in North America, is a random quantity. Thus uncertainty due to randomness can be better characterized by the collection of more data, while real variability will remain constant. In some papers [Burmaster, 1994], the term uncertainty has been reserved specifically for uncertainty that arises from randomness. Though uncertainty has been distinguished from variability, most previous works have not modeled each of these separately. They are usually grouped together and modeled using probability theory. Examples of such can be found in Hattis and Silver [1994], wherein the authors provide an illustration of inter-individual variability as determined in the uptake of a toxicant, systemic pharmacokinetic parameters, and the response observed.

A classification by Morgan and Henrion [1990] organizes the sources of uncertainty in empirical quantities into seven categories: 1) statistical variation; 2) subjective judgment; 3) linguistic imprecision; 4) variability; 5) inherent randomness of the system; 6) approximation, due to simplification of the model; and 7) disagreement amongst the experts. Statistical variation in measurements contributes to random errors indicating evidence focuses on disjoint events. Even though the type of uncertainty most relevant to this seems to be randomness, the inadequacy of the measurement device to precisely measure the property can also lead to evidence that focuses on a collection of disjoint events that are sets of numbers rather than a single number. While statistical variation can be considered to reflect the true property of the system, subjective judgments and inherent errors in instruments can result in systematic errors (deviation from the true value of the mean) that are properties of the measuring device. Subjective

judgments are also inherently vague and non-crisp indicating evidence focusing on ranges of numbers (sets or intervals) rather than single numbers (singletons). This leads to the next category, uncertainty due to linguistic imprecision, that is a result of the inability of the human mind to precisely define and communicate the occurrence of an event in crisp mathematical terms. This uncertainty, due to the nature of the expression of linguistic variables, results mostly in uncertainty that is vague. The next form of uncertainty, variability, in the same context as that defined in the earlier section, is the uncertainty arising due to the differences between individuals within a population. Such uncertainty can result in non-specific uncertainty due to the fact that within one population there are multiple values that are all true. For example if one is to determine the risks from exposure to a chemical X, multiple risk values are possible because the susceptibility of each sub-population to the chemical varies according to the characteristics of that sub-population (such as age, ethnicity, gender, etc). Hence one is ambiguous about the risk value. In contrast to variability, randomness inherent in the system is the true randomness exhibited by the system and hence as the term indicates, results in randomness. In addition to the randomness and variability in the system, uncertainty from approximations results in modeling errors and do not necessarily contribute to any one specific type of uncertainty. Finally, disagreement between experts is similar to acquiring uncertainty resulting from subjective judgments and can lead to uncertainty that is either fuzzy or ambiguous. As opposed to Morgan and Henrion's classification, which is more general, Bogen [1990] provides an in-depth classification of uncertainty according to the step at which it occurs in the risk assessment process. This classification differs from others in that Bogen tries to identify uncertainties arising at

each stage of health risk assessment instead of defining the character of uncertainty itself [Table 2-1].

Some other classifications of uncertainty include:

National Research Council [NRC, 1994] classification

1) Parameter uncertainty
   i) Random and systematic measurement errors resulting from the imprecision and defects in the measurement equipment;
   ii) Use of generic or surrogate data instead of analyzing the desired parameter directly;
   iii) Random sampling error arising from limited sampling;
   iv) Non-representativeness, a product of using wrong processes or data set for deriving the parameter of interest
2) Modeling uncertainty - which can account for uncertainty as high as a factor of 1000, results from the inappropriate choice of models.
   i) Inaccurate determination of the relationship between variables;
   ii) Assumptions and simplifications of reality;
   iii) Exclusion of one or more relevant variables;
   iv) Use of general variables to represent all the sites;
   v) Misrepresentation of the granularity of the model (due to incomplete understanding of the sub-components of the system).

Table 2-1: Classification of sources of uncertainty in risk assessment [Bogen, 1990]

Hazard Identification

- Unidentified hazards
- Definition of incidence of an outcome in a given study
- Different study results
- Different study qualities
  - Conduct
  - Definition of control population
  - Physical chemical similarity of chemical studied to that of concern
- Different study types
  - Prospective, case-control, bioassay, *in vivo* screen, *in vitro* screen, Test species, strain, sex, System, Exposure route, duration
- Extrapolation of available evidence to target human population

Dose-response assessment

- Extrapolation of tested doses to human doses
- Definition of 'positive responses' in a given study
  - independent vs. joint event
  - continuous vs. dichotomous input response data
- Parameter estimation
- Different dose-response sets
  - Results
  - Qualities
  - Types
- Model selection for low-dose risk extrapolation
  - Low-dose functional behavior of dose-response relationship (threshold, sublinear, linear, supralinear, flexible)
  - Role of time (dose frequency, rate, duration, age at exposure, fraction of life time exposed)
  - Pharmacokinetic model of effective dose as a function of applied dose
  - Impact of competing risks

Exposure assessment

- Contamination-scenario characterization (production, distribution, domestic and industrial storage and use, disposal, environmental transport, transformation and decay, geographic bounds, temporal bounds)
  - Parameter estimation error
  - Field measurement error
- Exposure-scenario characterization
  - Exposure-route identification
  - Exposure dynamics model
- Target-population identification
  - Potentially exposed populations
  - Population stability over time
  - Integrated exposure profile

Risk characterization

- Component uncertainties
  - Hazard identification
  - Dose-response uncertainty
  - Exposure assessment

Rowe's classification [1994]:

1) Temporal - uncertainty in future states and uncertainties in past states;
2) Structural - uncertainty due to complexity;
3) Metrical - uncertainty in measurement;
4) Translational - uncertainty in explaining uncertain results

In any parameter, the different types of the uncertainty may co-exist or one or more may dominate. Rowe further adds that all the above four classes of uncertainty are subject to variability, which arises due to the underlying variants and inconsistencies inherent in the system, and variation within the individual components of a system.

Finally, bias, which is a result of the systematic errors in measurements and subjective judgments, is a common cause of errors in models that rely on expert opinions. Subjective judgments contribute to bias due to their reliance on judgments of representativeness, availability, adjustment and anchoring heuristics [Tversky and Kahnemann, 1974], Representativeness is a heuristic used when a subject is asked to judge the probability that a specific event belongs to a certain class or process; availability is often employed when the expert is asked to determine the frequency of a class or the plausibility of a certain development; and, adjustment is often employed in numerical predictions when a relevant value is available to the subject.

All the sources of uncertainty identified above are shown in Table 2-2. As seen in the table, uncertainty is not limited to randomness and thus there is a need to adopt other formalisms. A formal introduction to these other formalisms along with probability theory is given in the following sections.

Table 2-2: **Sources of uncertainty and the corresponding type of uncertainty associated with the source**

Cause	Type
Morgan and Henrion's Classification	
Statistical variation	Mostly Random and Ambiguous
Subjective judgment	Random, Ambiguous, and Vague
Linguistic imprecision	Mostly Vague
Variability	Vague and Ambiguous
Inherent randomness of the system	Random
Approximation, due to simplification of the model	Modeling Error
Disagreement amongst the experts	Ambiguous
NRC Classification	
Parameter uncertainty	Random, vague and Ambiguous
Modeling uncertainty	Modeling Error
Rowe's Classification	
Temporal	Mostly Random
Structural	Vague and Ambiguous
Metrical	Random, Vague and Ambiguous, Also Bias
Translational	Vague

2.2 Methods of Uncertainty Analysis

Uncertainty has always been a dominant part of modeling, and its abstract character has resulted in controversial and conflicting views on the choice of the most appropriate model for representing it. There are at least two main branches of methods, probabilistic and non-probabilistic, distinguished axiomatically, and these are subdivided into five theories, distinguished by their context for modeling uncertainty; 1) Frequentist probability theory; 2) Bayesian probability theory [Savage, 1961]; 3) Evidence theory [Shafer, 1976]; 4) Possibility theory [Zadeh, 1972; Dubois and Prade, 1988]; and 5) Fuzzy set theory [Zadeh, 1965]. Axiomatically, probabilistic methods differ from non-

probabilistic methods in one axiom, in that, probabilistic models require strict additivity, while non-probabilistic models do not require an adherence to this axiom. Conceptually, probability theory is used to characterize chance and randomness, and the quantified belief measures are supposed to precisely represent the occurrence of the states of the world. In contrast to the assumption of chance and completeness of data by probability theory, evidence theory is based on the assertion that "since uncertainty arises from our incomplete knowledge of the world, the belief measures should reflect the analysts' belief of occurrence of the state of the world such that any ignorance indicating a lack of knowledge should also be represented in the uncertainty analysis." In contrast to these two theories, fuzzy set theory characterizes uncertainty that arises due to the vagueness in the definition of a set itself (such as the definition of the set - 'high risk').

Among the various probabilistic theories, Frequentist probability theory has been the most common method, followed by Bayesian theory applied mostly in the field of management where subjective judgments are more easily accepted. Non-probabilistic theories, such as, evidence theory, possibility theory and fuzzy set theory have also been applied in various fields with a great deal of success; however, the non-probabilistic nature and unfamiliarity of these theories has been a source of hesitation by many individuals in accepting them in models. Though the proponents of the non-probabilistic theories claim that uncertainty is of many types (randomness, vagueness and ambiguity), and hence is best modeled by formalisms that are more flexible in absorbing all the types, the probabilistic community has expressed it's caution in using them as an effective way of representing uncertainties [Cheeseman, 1986; Wu *et al*., 1990; Laviolette and Seaman 1994; Laviolette *et al.*, 1995]. As more sources of uncertainty are entertained, however,

it will become imperative to combine the probabilistic and non-probabilistic models [Klir, 1994; Zadeh, 1995].

2.2.1 Probability Theories - Frequentist and Bayesian

From the 17th to the early 19th centuries, uncertainty was usually associated with randomness of systems and thus was defined solely on a probability space. A probability space, defined on a domain Ω and the σ-algebra $\Im$ on Ω, is given by the probability triple $(\Omega, \Im, \mathrm{P})$, where P is a probability measure defined on $\Im$. The σ-algebra $\Im$, is defined as a collection of subsets of Ω, which satisfy the following properties:

1. $\Omega \in \Im$ **(2-1)**

2. If a subset $A \in \Im$, then, $A^c = \{\omega : \omega \notin A\} \in \Im$ **(2-2)**

3. If $A_i \in \Im; i = 1,2,....$, then, $\cup A_i \in \Im$ **(2-3)**

Now, for the element A of $\Im$, where elements A are considered as events, probability in the classic sense is required to satisfy the following Kolmogorov's axioms:

Axiom 1 **(Non-negativity):** $0 \le P(A) \le 1$ **(2-4)**

Axiom 2 **(Normalization):** $P(\Omega) = 1$ **(2-5)**

Axiom 3 (**Finite Additivity):** If $A_i \in \Im, i = 1,2,...,n$ are disjoint, i.e.,

$A_i \cap A_j = \varnothing$, then, for all $i \neq j$

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) \qquad \textbf{(2-6)}$$

Within the period of development of probability theory, two schools of thought, Bayesian and Frequentist, originated from how probability was viewed. The major difference being, for one (Bayesians), probability is the degree of confirmation about the occurrence of an event while to the other (Frequentists), probability is the frequency of occurrence of an event in the long run. This subtle difference has a significant impact on how uncertainty is modeled and how the probability estimates are enumerated. In the simplest case, a typical Frequentist analysis consists of *n* observations of a certain random variable, X, and the parameters of X are determined from the observations using methods such as maximum likelihood estimation or method of moments. The entire process of enumerating the parameters is strictly dependent on direct observational data and thus probabilities are considered to be derived objectively. In contrast to this analysis, Bayesians accept subjective judgments as a natural substitute for the lack of direct observational data and believe that the probabilities are degrees of belief deliberately constructed and adopted on the basis of all the available evidence, both objective and subjective.

The three basic characteristics of the Bayesian theory are [Pearl, 1987]:

1. The reliance on a complete probabilistic model of the domain;
2. Willingness to accept subjective judgments as an expedient substitute for empirical data; and
3. The use of Bayes' conditionalization as the primary mechanism for updating beliefs in light of new information.

Mathematically, the Bayesian method is based on Bayes' theorem of conditionalization:

$$P(H_i|e) = \frac{P(e|H_i)P(H_i)}{P(e)}, \quad \textbf{(2-7)}$$

stating that the belief accorded to a particular hypothesis H_i upon obtaining evidence *e* can be computed by conditioning one's prior belief $P(H_i)$ and the likelihood that *e* will materialize assuming H_i is true. More formally, $P(H_i)$, also known as the prior, is the probability defined over the set of all hypothesis $H_i \in \Im$, $i = 1,2,....,n$, and $P(e)$ is the total probability of evidence given that each hypothesis H_i is true. In other words, if the sample space is partitioned such that each H_i forms a partition and if e is the evidence manifested, then,

$$P(e) = \sum_{i=1}^{n} P(e|H_i)P(H_i) \quad \textbf{(2-8)}$$

$P(e \mid H_i)$ is the likelihood that the evidence will be observed for when each H_i is true. The degrees of belief thus accorded are required to satisfy the following two rules: the axiom of additivity, for any H,

$$P(H|e) + P(\overline{H}|e) = 1, \quad \textbf{(2-9)}$$

and the product rule, which states that for any two events H_1 and H_2,

$$P(H_1, H_2|e) = P(H_2|H_1, e)P(H_1|e) \quad \textbf{(2-10)}$$

It is the dependence of priors on expert judgments and the conditional probabilities on the right hand side of Equation 2-7 that leads one to identify the subscribers of Bayesian

theory as subjectivists. Many other formalisms of the Bayes' theorem [Shafer and Tversky, 1985] and a plethora of articles discussing the subjective nature of Bayesian theory are available. An elaboration of the Bayes' theorem from an environmental perspective is presented in the papers by Hill [1996] and Smith [1996].

Due to the familiarity and objectivism associated with Frequentist analysis, most of the analyses in human health risk assessment use this method. The Frequentist method is used in the derivation of dose-response relationships for carcinogens, reference doses for non-carcinogens, and the determination of concentration profiles at various hazardous waste sites [EPA, 1989]. Dose-response analysis is usually conducted using either statistical or mechanistic models (Table 2-3). The statistical models are based on the premise that individuals have a certain tolerance and, hence, they model the tolerance each individual exhibits to a chemical; the tolerance level representing the point above which an individual will develop an adverse effect beyond a certain predefined background level. Probit, Logit and Weibull are common statistical models. The Probit model is based on the assumption that the log tolerances are normally distributed with mean μ and standard deviation σ, while Logit is a Probit with log of the doses. Mechanistic models assume that a positive response is the result of a random occurrence of one or more biological events. One-hit model (from radiation) is based on the premise that a random exposure to one molecule of an agent is sufficient to trigger the biological event, conversely, a multi-hit model assumes that at least k hits have to occur before the chemical can induce an irreversible damage at the target site. The Armitage and Doll model assumes that more than one biological event has to take place before an irreversible damage can be manifested in a tissue [Armitage and Doll, 1954]. This

model can accommodate a quadratic term in the equation thereby allowing the modeling of curvature in a dose-response curve. Moolgavkar and Knudson (MKR) [1981] extend this model to include the initiation and promotion stages of the carcinogenesis process.

Table 2-3: Common probability response functions used in risk assessments

Model	*Function for Probability of a response*
Probit	$(2\pi)^{-1/2}\int_{-\alpha}^{\alpha+\beta\log d}\exp(-u^2/2)du; \quad \beta > 0$
Logit	$[1+\exp(\alpha-\beta\log d)]^{-1}$
Weibull	
One-hit	$1-\exp(-\beta d)$
Multi-hit	$\int_0^d \left[\frac{[a^k][s^{k-1}][\exp(-as)]}{\Gamma(u)}\right]ds$ *where,* $\Gamma(u)=\int_0^\infty s^{u-1}\exp(-s)ds$
Multi-stage	$1-\exp\{-[q(0)+q(1)d+q(2)d^2+....+q(k)d^k]\}$ *k is the number of biological stages required in the carcinogenic process*
MKR	$l(t)=\mu_1\mu_2\int_0^t X(s)\exp([(\alpha_2-\beta_2)(t-s)])ds$ *l(t) – incidence of tumors* μ_1 *-rate of first mutation* μ_2 *-rate of second mutation (mutation of transformed cells* *X(s) – Number of normal cells in the target organ at time s* α_2 *-birth rate of new cells* β_2 *-death rate of cells*

All the dose-response models (such as probit, weibull, linear multistage, etc.), reflect the use of Frequentist methods in that the response probabilities are derived from functions that are fit to strict observational data in the form of single scalar values (singletons),

however, the claim of strict objectivity by the Frequentists is questionable as the inputs to the choice of the experiment and the selection of available distributions to fit the data are determined subjectively from an expert [Savage, 1976; Evans *et al*., 1994].

As opposed to the Frequentist models, Bayesian models have found limited application. Although the application of Bayesian methods in the fields of artificial intelligence and probabilistic risk assessment of nuclear power plants has been studied extensively, research regarding their use in health risk assessment has started gaining momentum in the past two decades. Some attempts have been made in the past to implement Bayesian approaches in dose-response assessments - the need to combine data from various experiments has motivated DuMouchel and Harris [1983] to develop a Bayesian statistical model for interspecies extrapolation of dose-response functions and, a Bayesian approach has also been used by EPA's (Environmental Protection Agency) Office of Air Quality to determine non-cancer health effects of air pollutants [Whitfield and Wallsten, 1989]. Another application has been the use of a Bayesian Monte Carlo method for updating uncertainty in integrated environmental health risk assessment [Brand and Small, 1995]. And more recently, Hasselblad and Jarabek [1996] have comprehensively demonstrated through a number of case studies, the effectiveness of the Bayesian approach in modeling the dose-response behavior of toxic chemicals. With the increasing recognition for the use of expert judgments to fill in data gaps, there has been an increasing awareness in the human health risk assessment community of the use of Bayesian methods.

Though Bayesian and Frequentist theories differ in how they derive probabilities they both must satisfy Kolmogorov's axioms (Equations 2-4 to 2-6). The third axiom,

also called the law of additivity, states that the sum of the probability of occurrence and that of non-occurrence should be unity. In other words, information obtained to determine the occurrence of a given event also asserts the probability of its non-occurrence. This implies two things: 1) the events have to be crisp (the boundaries between events are crisp and unambiguous); and 2) there is no ignorance (ignorance being, if we have knowledge of the occurrence of an event, we do not necessarily know anything about it's non-occurrence, i.e., we are ignorant about it's non-occurrence). This, however, is not always satisfied when there is a lack of complete evidence or sometimes we don't know how the events are defined. For example, in a dose-response relationship, there is seldom information to completely characterize the effect of a chemical at a certain dose. To precisely state that the response to a certain dose can occur 20 times out of 100 and stating that the probability in this case is 0.2 is assuming more information, because in probability one automatically assumes that response of non-occurrence at this dose is, $q=1-p=0.8$. This is further magnified when the dose-response curve is extrapolated to extremely lower doses because the upper bound of the initial dose-response slope is what is extrapolated to the lower doses. In addition, the events for which probability is sought are themselves not crisply defined. This could be because of the lack of clear definition of the end points. In the case of fatalities, the response is easier to define. When the response being measured is the degradation of a certain human function, however, it is difficult to precisely pinpoint when the "response" (or degradation) has occurred. The inability of probability theory to model non-crisp event boundaries and ignorance has led to development of the non-probabilistic theories which will be discussed in the next few sections.

2.2.2 Non-Probabilistic Theories in Risk Assessment

Among the non-probabilistic theories, evidence theory and fuzzy set theory are the predominant ones that have found applications in risk analysis [Krause and Fox, 1995; Lein, 1992; Bardossy *et al*., 1993].

2.2.2.1 Evidence theory

Evidence theory (also known as Dempster-Shafer Theory), originally conceived by Dempster [1968] to determine the lower and upper probabilities and later extended by Shafer [1976] to include the epistemic degrees of belief, consists of assigning probabilities on the basis of the belief in evidence obtained to support a proposition. Therefore, belief is assigned to individual pieces of evidence, and is not considered to yield total probabilities directly. In this theory the elements of a finite universe of discourse and its subsets are identified and then beliefs are assigned to each one of these subsets. The universe of all the subsets is usually referred to as the frame of discernment and the subsets for which the assigned belief is non-zero are called the focal elements. Beliefs are assigned not only to the subsets, but also to all the individual elements of the universe. For any subset $A \in \Omega$, if we have a function, $m : \wp(\Omega) \to [0,1]$, which is called a basic probability assignment (bpa), where $m(\varnothing) = 0$ and $\sum_{A_i \in \Im} m(A_i) = 1$, then the basic probability assignment $m(A)$ represents the belief one commits exactly to A, and not the total belief committed to A, i.e., not to any other subsets of the universe. A set F, of all

elements for which $m(A) > 0$ is called the set of focal elements or the focal set, i.e., $F = \{A \subseteq \Omega : m(A) > 0\}$, and $F \subseteq P(\Omega)$. The total belief committed to A is determined by the following equation,

$$Bel(A) = \sum_{A_i \subseteq A} m(A_i) \qquad \textbf{(2-11)}$$

In contrast to the rule of additivity as given by Equation 2-6, the axiom of belief functions states that for every positive integer *n* and every collection $A_i, i = 1,2,....,n$ of subsets of Ω,

$$Bel(\bigcup_{i=1}^{N} A_i) \geq \sum_{i} Bel(A_i) - \sum_{i<j} Bel(A_i \cap A_j) + ... + (-1)^{n+1} Bel(A_1 \cap .... \cap A_N)$$

(2-12)

Equation 2-12 states that the belief measures of the union of *N* subsets does not have to be equal to the sum of individual subsets; the difference being indicative of the presence of ignorance in the assignment of beliefs to each subset. This concept is more evident when one considers only two subsets, *A* and $\overline{A}$. Since $Bel(A \cup \overline{A}) = 1.0$, according to Equation 2-12,

$$Bel(A) + Bel(\overline{A}) \leq 1 \qquad \textbf{(2-13)}$$

The difference between 1 and ($Bel(A) + Bel(\overline{A})$) represents the ignorance of an analyst to precisely assign beliefs to the negation of A and thus allows him to model uncertainties more realistically. Now, if $Bel(\overline{A})$ represents the extent to which one believes in the negation of *A*, then the dual of a belief measure, called the plausibility measure, *Pl(A)*, is

given as,

$$Pl(A) = 1 - Bel(\overline{A}) = \sum_{A_i \cap A \neq \varnothing} m(A_i) \qquad \textbf{(2-14)}$$

Similar to the belief functions, plausibility measures satisfy the following condition,

$$Pl(A) + Pl(\overline{A}) \geq 1 \qquad \textbf{(2-15)}$$

Above equations represent the fundamental difference between evidence theory and Bayesian theories. In evidence theory, for every collection of sets, there are two measures to quantify the uncertainties. By allowing a quantitative determination of two measures for a single event (belief and plausibility) evidence theory permits one to set bounds for the occurrence of an event. The belief and plausibility measures for an event can be viewed as the lower and upper bounds, respectively, of the probability measure of that event. When the evidence is specific to singletons, belief and plausibility measures reduce to probability measures.

Similar to Bayesian theory, evidence theory can also combine knowledge from various sources. Dempster's rule of combination is used to combine the evidences from two sources, m_1 and m_2 specified on Ω, and $B, C \subseteq \Omega$, as,

$$m_{1,2}(A) = \frac{\sum_{B \cap C = A} m_1(B) \cdot m_2(C)}{1 - K} \qquad \textbf{(2-16)}$$

for $A \neq \varnothing$, where

$$K = \sum_{B \cap C = \varnothing} m_1(B) \cdot m_2(C) \qquad \textbf{(2-17)}$$

The term 1-K is a normalization factor used to discard any information that belongs to the null set in the intersection set of m_1 and m_2. Unlike Bayes' rule, Dempster's rule of combination does not condition priors over the new information, rather prior and new information are combined equally. A more in-depth analysis of Equation 2-16 can be found in Shafer's treatise on Theory of Evidence [Shafer, 1976].

Over the years, evidence theory has been widely used in the design of medical diagnostic systems, however, its use in human health risk assessments has been very limited. Evidence theory offers an excellent framework for the inclusion of qualitative information into a model, or when information flow is such that judgments are made from a superset to the subsets (from general to specific). Krause *et al*. [1995] explore the idea of using belief and plausibility measures in deriving plausible conclusions about the risks from carcinogenic chemicals. They consider the inputs to the expert system as being supplied by the expert as linguistic qualitative arguments and are only defined by weak quantitative measures.

2.2.2.2 Fuzzy Set Theory

Fuzzy sets, introduced by Zadeh in 1965 [Zadeh, 1965], have been used extensively in control, optimization and decision making systems. Fuzzy logic, in comparison to classical binary logic, "exploits the tolerance for imprecision" in systems and allows for partial truths to be assigned to various states. Instead of the truth values being in either the 0 (false) or the 1 (true) state, a partial truth within the interval [0,1] is used. In conventional set theoretic formulations, a given element x, either belongs to the

set A or to the complement $\overline{A}$. Therefore, for a given set A,

$$Truth(x) = \begin{cases} 1 \text{ iff } x \in A \\ 0 \text{ iff } x \notin A \end{cases} \quad \textbf{(2-18)}$$

In contrast to this hard classification, fuzzy set theory introduces a softer concept in terms of a membership function, which determines the degree to which the element x belongs to the set A. Hence, instead of assigning 0 or 1, a value in the range [0,1] can be assigned. The supporters of fuzzy set theory use this concept to support their argument that fuzzy set theory can model linguistic imprecision more accurately than conventional probabilistic models. Linguistic imprecision is usually reflected in vague or ambiguous statements such as "likely", "seldom", "more or less", and so on. Fuzzy logicians claim that a range of values is a better way of representing such variables. A fuzzy set, $\tilde{A}$ is given by a membership function, $\mu_{\tilde{A}}(x)$, on a real line [0,1], and represents the degree to which an element, x belongs to the fuzzy set $\tilde{A}$.

A fuzzy set which can be discrete or continuous, can be represented as

$$\tilde{A} = \left\{ \frac{\mu_{\tilde{A}}(x_1)}{x_1} + \frac{\mu_{\tilde{A}}(x_2)}{x_2} + \ldots\ldots \right\} = \sum_i \frac{\mu_{\tilde{A}}(x_i)}{x_i} \quad \textbf{(2-19)}$$

The horizontal bar in the above equation is not a quotient but a delimiter, and the "+" sign is the union and the $\sum$ indicates the collective union of all the membership terms.

Membership functions and probability distribution functions should not be confused with each other, as the former deals with partial degree in assignment of an element to a particular set, while the latter deals with relative likelihood. The

implications of using membership functions to represent a variable are discussed more in the next section.

Once a membership function for a variable is defined, two principle methods exist to perform fuzzy arithmetic. Zadeh has proposed an extension principle to functionally map the products of elements from two discrete universes U_1 and U_2 to another discrete universe as,

$$f(\tilde{A}) = \left\{ \sum \frac{\min\left[\mu_1(u_1), \mu_2(u_2)\right]}{f(u_1, u_2)} \middle| u_1 \in U_1, u_2 \in U_2 \right\} \quad \textbf{(2-20)}$$

and in the case where more than one of the combinations of the input variables, U_1 and U_2, are mapped to the same variable in the output space, V, the maximum membership grades of the combinations of the discrete mappings to the same output variable are taken:

$$\mu_{\tilde{A}}(v) = \max_{v=f(u_1,u_2)} \left[\min\left\{\mu_1(u_1), \mu_2(u_2)\right\}\right] \quad \textbf{(2-21)}$$

While rigorous, the extension principle can be difficult to implement for many functions. An alternative method is based on interval analysis, wherein the membership function is divided into a number of λ-cut levels and standard interval analysis operations can be carried out on the λ-cuts. A λ-cut set, A_λ, where $0 \le \lambda \le 1$, is a crisp set such that, $A_\lambda = \left\{x \middle| \mu_{\tilde{A}}(x) \ge \lambda\right\}$. Figure 2-1 shows an example of the membership function and the λ-cut sets at a membership of 0.3 and 0.6. The sets represented by the intervals [$x_{1(0.3)}$, $x_{2(0.3)}$] and {$x_{1(0.6)}$, $x_{2(0.6)}$] define the λ-cut sets $A_{0.3}$ and $A_{0.6}$.

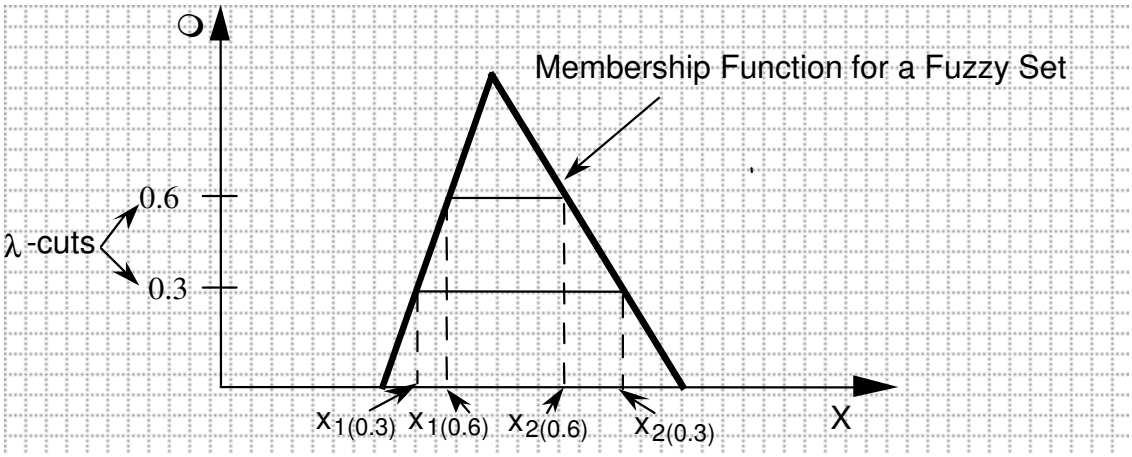


Figure 2-1 Membership function and λ-cut illustration.

A number of other specific operations depending on the field of application (such as, rule-based systems, fuzzy decision making, pattern recognition, etc.,) have been developed (see Ross, 1995).

Fuzzy set theory has had limited application in human health risk assessments. Bardossy *et al*. [1993] use fuzzy non-linear regression analysis in the determination of dose-response relationships of N-nitroso compounds. In this model, the fuzzy parameters for the dose response relationship are derived using fuzzy non-linear regression. Ferson and Kuhn [1992] explore the use of fuzzy arithmetic in propagating uncertainty in ecological risk assessment. Due to the acknowledgment and higher acceptance of vagueness in decision-making, there have been more applications of fuzzy set theory in the risk management of hazardous waste sites than in risk assessments [Anandalingam and Westfall, 1988; Lee *et al*., 1994; Ross and Donald, 1995; Donald and Ross, 1996].

2.2.3 Discussion of the Various Methods

As can be seen from the previous sections, each method has a unique advantage over the others and also, more uncertainty is represented as the method accommodates a range of numbers as opposed to point estimates. In the Frequentist method, the assigned beliefs are considered to be the property of the world and not the property of the observer who is assigning the beliefs. The method depends more on empirical data as determined by a statistical experiment, therefore the derived probabilities are not affected by subjective bias. The purely objective nature of this theory has, however, been questioned on the basis that the choice of statistical experiments, the inclusion or exclusion of data, the selection of distributions, and the choice of method for estimating parameters, all include an input of the expert's understanding of the system. But mostly, this method is widely accepted as being more objective than subjective in nature. Bayesian theory, on the contrary, offers a framework to include both Frequentist concepts and expert judgment to estimate the priors, and also enables an updating of these priors depending on the current state of knowledge of the system. For example, a system may consist of various components whose failure might lead to a change in the probability estimates of the other components. Consider the case where a hazardous chemical spill has occurred. Such a situation which might arise due to a number of reasons can affect the prior probabilities assigned to the probability of exposure ($P(Exposure|spill)$). Bayesian theory thus provides the analyst a framework to update the probabilities with acquisition of more data. Bayesians' claim that probability is the observer's state of knowledge of a system at a given point, and it allows an expert to use his judgment in providing

probabilities. In cases where complete data to define a distribution is lacking it would be prudent to accept such an expert's judgments to fill in the data gaps and derive the estimates of probability. Consequently, with the availability of additional data the inferences about an estimate can be changed and updated according to Bayes' theorem.

Bayesian theory, however, with an adherence to the axiom of additivity, assumes that the probabilities of an event occurring are known with certainty, i.e., if we determine that an event *A* can occur with a probability, *p*, then the probability of the event not occurring is *1-p*, even though there may be no evidence to support this or, worse, there may be evidence to refute this. In other words, the evidence (objective or subjective) is considered to be accurate in determining the probability of occurrence of an event.

The Dempster-Shafer theory disagrees with this Bayesian concept in that the belief of an event *A* occurring, *p*, does not mean that the belief of non-occurrence is *1-p*. Shafer [1990] points out that "instead of thinking of a chance model in terms of the facts, we should think in terms of a chance model for the reliability and the meaning of our evidence." Therefore, we do not have to be committed completely to any one estimate in the absence of relevant data. Instead belief values can be assigned to the subsets of the universal set. For example, let Ω = {cancer by ingestion (CIG), cancer by inhalation (CIN), cancer by dermal absorption (CDA), no cancer}, represent the frame of discernment. If I believe from available information that an individual exposed to a contaminant will develop cancer by ingestion is 0.002, this does not mean that the individual not contracting cancer by this exposure route is 0.998. Table 2-4 shows the basic probability assignments *m(A)* for the rest of the subsets of this example problem domain. Equations given in section 2.2.2.1 are used for the determination of the belief

and plausibility measures. In the Dempster-Shafer theory, an overlap of the information is allowed and it is not necessary to commit to either one subset. Such representation, hence, includes the ignorance of knowledge and denotes a range of possible values. In practice, representations such as these are not uncommon occurrences. An expert or the observer assigning the belief values is usually not sure (in doubt) about the commitment to any one proposition, rather, the expert spreads his belief to more than one proposition. In evidence theory, thus, two sets of measures, belief and plausibility, are calculated and then assigned to each subset.

In fuzzy set theory, instead of assigning crisp probabilities to the occurrence of cancer for a certain route, a range of numbers defined by a function can be used. The rationale for representing uncertainty in such fashion can be justified on the basis that experts are more comfortable in expressing themselves linguistically over a range of numbers. Usually these functions have been limited to trapezoidal and triangular shapes. For example, a triangular membership function given over a certain range (Figure 2-2) can be used to represent an expert's estimate of the probability of fatality from cancer due to inhalation of some toxic gas to be 'around 0.2'. Similar membership functions can be defined for other variables and fuzzy arithmetic or interval analysis can be used to propagate the uncertainties over the probability space.

Table 2-4: Example Belief and Plausibility measures

Focal Elements A_i	$m(A_i)$	$Bel(A_i)$	$Pl(A_i)$
Cancer by Ingestion (CIG)	0.006	0.006	0.197
Cancer by Inhalation (CIN)	0.003	0.003	0.193
Cancer by Dermal absorption (CDA)	0.000	0.000	0.183
CIG ∪ CIN	0.009	0.018	0.219
CIG ∪ CDA	0.002	0.008	0.198
CIN ∪ CDA	0.001	0.004	0.195
CIG ∪ CIN ∪ CDA	0.11	0.130	0.200
No Cancer	0.800	0.800	0.870
CIG ∪ No Cancer	0.000	0.806	0.996
CIN ∪ No Cancer	0.000	0.803	0.992
CDA ∪ No Cancer	0.000	0.800	0.982
CIG ∪ CIN ∪ No Cancer	0.000	0.818	1.000
CIG ∪ CDA ∪ No Cancer	0.000	0.808	0.997
CIN ∪ CDA ∪ No Cancer	0.000	0.804	0.994
CIG ∪ CIN ∪ CDA ∪ No Cancer	0.069	1.000	1.000

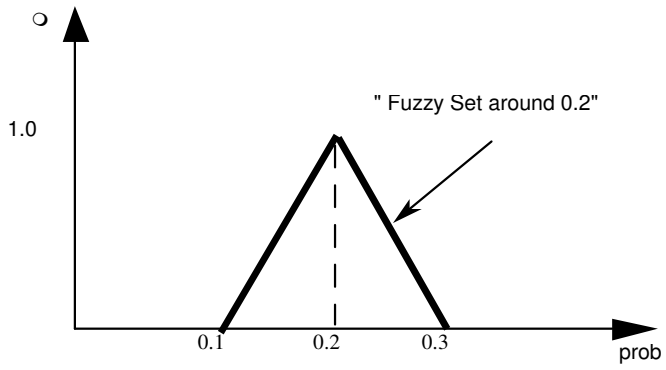


Figure 2-2: A triangular membership function representing the variable 'around 0.2'.

Another theory, which has not had much application in risk assessments, but of equal importance in modeling uncertainty, is possibility theory [Dubois and Prade, 1988]. Possibility measures can be interpreted in terms of two concepts: i) Zadeh's concept of a fuzzy set, where the possibility distribution function, $\pi(x) = \mu_A(x)$; and ii) evidence theory, when the belief measures are consonant, *i.e.*, the focal elements are nested such that $A_1 \subset A_2 \subset \ldots\ldots \subset A_n$. Possibility distributions have also been identified with distributions defined on probabilities [Tanaka *et al.*, 1983; Misra and Weber, 1989, 1992]. Possibility distributions defined in such terms are merely fuzzy sets on the crisp probability values. The historical interpretations based on fuzzy sets, however, should be differentiated from the possibility distributions acquired from assuming consonance of intervals (nested intervals), in which case, the distribution is a set-based contour function of plausibilities (from Evidence theory). Under information constraints, imprecision in the probabilities are unavoidable, hence, this method (Dubois and Prade, 1988) of representation for possibility theory seems more logical and realistic [Klir, 1994]. Representing uncertainty using possibility theory, of the Dubois and Prade type, is the core of this dissertation and is presented further in Chapter 3.

2.3 Inclusion and Propagation of Uncertainty

Uncertainty measures derived by using any of the above concepts can be characterized either as a point estimate, as an interval, or as a distribution. Propagation techniques should be appropriate to the form of characterization. In point estimates, the associated uncertainty is usually summarized in a single value using either an uncertainty

factor (non-cancer risk assessment), or 95 percent upper bounds of values derived from linear extrapolations (cancer risk assessment) or by considering the upper percentile values (exposure assessment). On the other hand, in the case of intervals or distributions, ranges of numbers represented either as both the lower and upper bounds, or as a functional distribution over the variable space are used to characterize uncertainty. While simple mathematical operations can be used to propagate uncertainty represented as single numbers, propagation of intervals and distributions through a risk assessment model warrant special operations. In probabilistic methods, there are at least four methods used for propagating uncertainty represented by distributions [Cox and Baybutt, 1981]: i) analytic techniques; ii) differential sensitivity techniques; iii) Monte Carlo simulation; and iv) response surface approaches. Of the available methods, Monte Carlo simulations are the most widely used in the field of risk assessment [Thompson *et al.*, 1992, Smith, 1994, Hoffmann and Hammonds, 1994]. The complexity of the analytic techniques and differential sensitivity techniques increases with the number of variables and hence these methods have not been used much in the propagation of uncertainty, especially in human health risk assessments. An in-depth explanation of the use of analytic methods to propagate large errors is presented in a paper by Seiler [1987] (see also Morgan and Henrion, 1990 for a description and comparison of various methods). Interval methods are used to propagate uncertainty represented by intervals, those that have been derived from either probabilistic concepts or non-probabilistic concepts. Extension principle is another common method used for propagating uncertainty represented as fuzzy membership functions [Ross, 1995]. Propagation by the extension principle is dependent on the discretization of the membership function and can therefore

yield extraneous outputs in coarsely discretized functions. This section focuses only on the most common methods of characterization and propagation of uncertainty in human health risk assessment.

2.3.1 Science-Based Uncertainty Factors

Conventionally, uncertainties arising from insufficient knowledge of animal-to-human extrapolations were accounted for by using factors that had a more numerical than mechanistic basis. Introduced by the Food and Drug Administration (FDA) to account for interspecies variability and the presence of sensitive sub-populations [Lehmann and Fitzugh, 1954], safety factors, also known as uncertainty factors, are the prescribed modes of handling uncertainty in non-cancer risk assessments. They are defined for high-to-low dose, inter- and intra-species and Low Observed Adverse Effect Levels (LOAEL)-to-No Observed Adverse Effect Levels (NOAEL) extrapolations.

Table 2-5: Uncertainty Factors used at various extrapolation levels

Extrapolations	Rationale	Default Uncertainty Factor
Variation within the general populations	Intended to be protective of sensitive subpopulations such as elderly and children	10
From animals to humans	Accounts for interspecies variability	10
Sub-chronic effects to chronic effects	Includes the uncertainty due long term toxic effects of the chemical	10
LOAEL[1]s to NOAEL[2]s	Accounts for uncertainty in how low a NOAEL is from LOAEL	10

[1]Low-Observed Adverse Effect Levels
[2]No-Observed Adverse Effect Levels

Application of an uncertainty factor (UF) is criticized on the basis that the validity of a NOAEL itself is questionable, since it is dependent on the number of animals tested, as 0-out-of-10 and 0-out-of-1000 have clearly different interpretations. And also, the actual conservatism attached to a reference dose derived using an uncertainty factor is arguable as the slope of the dose-response curve is not considered in the selection of the UF [Cornfield *et al.*, 1980] (more recently, a benchmark dose (BMD) approach that accounts for the slope of the dose-response curve has been developed to enhance the validity of deriving the NOAELs [EPA, 1995]). Table 2-5 shows the default uncertainty factors (UFs) as prescribed by the EPA for determining the reference doses (RfD) (EPA, 1989). The values from Table 2-5 have been replaced by factors that have been derived from the increased understanding of the mechanism of action of various chemicals, inter- and intra-species sensitivity, and acquisition of more data has led to the development of UFs that were more justifiable on a mechanistic basis. To counter the criticisms of lack of scientific basis for conventional safety factors, science-based uncertainty factors were developed. Dourson *et al*. [1996] highlight three approaches that have been developed for deriving science-based uncertainty factors: 1) Lewis-Lynch-Nikiforov's (LLN) model [1990]; 2) Renwick's Model [1993]; and 3) Probabilistic approaches [Swartout *et al.*, 1994]. The LLN model accounts for uncertainty in extrapolations from animal to humans by adopting adjustment factors as given in Equation 2-22 (see Table 2-6 for a description of terms). The magnitude of the adjustment factors are determined by the risk assessor and they vary according to the form of extrapolation and the strength of data supporting a cause-effect relationship.

$$No\ Effect\ Level\ in\ Human\ =\ \frac{NOAEL\ \ in\ Animal \times S}{I \times R \times Q_1 \times Q_2 \times Q_3 \times U \times C} \qquad \textbf{(2-22)}$$

Renwick's model is mainly applicable to intra- and interspecies extrapolations. Uncertainty factors derived from Renwick's model are based more on mechanistic information such as toxicokinetic and toxicodynamic differences between the species. Based on a study of toxicokinetic and toxicodynamic variations between and within species, Renwick recommends an uncertainty factor of four for toxicokinetic differences and a factor of 2.5 for toxicodynamic differences.

And finally probabilistic approaches are based on the rationale that enough data exists to specify a probabilistic distribution over the range of uncertainty factors [Swartout *et al.*, 1994]. The most likely distributions for each of the uncertainty factors is a log-normal, with the following parameters: 1) an UF of 10 represents the 95th percentile; 2) an UF of 3 represents the 50th percentile; 3) the subchronic-to-chronic and LOAEL-to-NOAEL UF is bounded by values of 1 and 50; and 4) the interspecies extrapolation UF is bounded by values of 0.2 and 50. Risk estimates derived from the use of the default safety factors are usually much higher than the actual values, the extent of conservatism increasing substantially with the number of input variables [Dourson and Strata, 1983]. In complex risk assessment models, where the number of variables is very large, this conservatism can yield risk values that are much larger than the actual realizable value. Additionally, the use of UFs to derive reference doses also suffers from the drawbacks of influencing an analyst to subsequently accept the value as the truth, thus limiting any search for better values.

Table 2-6: Adjustment factors for the LLN Model [Dourson *et al.*, 1996]

AF[a]	Description	Range of Values	Most Likely Value	Default Value
S	Scaling Factor to account for known quantitative differences between species and, the differences in experimental conditions and those likely to be encountered by humans	>0	NS[b]	1
I	Intraspecies variability	1-10	1-3[c]	10
R	Interspecies extrapolation	>0-10	NS	10
Q_1	Degree of Certainty that the critical effect observed in laboratory animals is relevant to humans	0.1-1		1
Q_2	Subchronic to chronic extrapolation	1-10	1-3	10
Q_3	LOAEL to NOAEL extrapolation	NS	2	10
U	Accounts for residual uncertainty in estimates of S, I, and R	1-10	NS	10
C	A nonscientific, judgmental safety factor	1-10	≤3	1

[a]Adjustment Factor
[b]Not stated in the LLN paper
[c]Based on high quality data

A second form of uncertainty representation that is being pursued by the EPA is to base the derivation of toxicity factors of carcinogenic agents on the 95% confidence interval (CI) bounds. The existing EPA guidelines [EPA, 1989] recommend a slope factor approach to estimate the risks from carcinogens; slope factor being defined as the upper 95% confidence interval bounds of the slope of the dose-response curve obtained from extrapolating from high- to low-doses. The guidelines also recommend the use of the upper 95th percentile values for the parameters (such as inhalation rate, ingestion rate, magnitude of contamination detected at the site, etc.) used in the assessment of the

exposure dose. This however has been shown to lead to an overly conservative estimate of risk.

Bogen [1994], Burmaster *et al*. [1993] and Finkel [1994] illustrate three different perspectives on the effect of compounding conservatism arising from using upper percentile values in their papers. Bogen determines the effect of using the upper *p*-fractile estimates as the input variates in multiplicative models. Let X_i represent the input variates and R the estimate of the predicted risk. If $O_p(X) = \log (X_p/X_{1-p})$, where $0.5 \le p \le 1$, and *p* is the upper *p*-fractile, the factor by which the sum of *p*-specific orders of uncertainty magnitude associated with the *n*-input variates overestimates the corresponding number associated with the input risk is given by Equation 2-23:

$$\beta = \frac{\sum_{i=1}^{n} O_p(X_i)}{O_p(R)} \qquad \textbf{(2-23)}$$

If the overconservatism factor, κ is expressed as,

$$\kappa = \frac{1-p}{1-p'} \qquad \textbf{(2-24)}$$

and $p' = \Phi(\beta\Phi^{-1}(p))$, where Φ and Φ^{-1} are the standard normal cumulative probability distribution function and its inverse respectively, and if the inputs have a lognormal distribution, with *p*=0.95, the overconservatism factor κ is 5.0, 99.7, and 506,000 when n is 2, 4, and 10, respectively [Bogen, 1994]. The overconservatism thus increases by significant amount with the number of input variates. Similarly, Burmaster *et al*. [1993] have determined as given by Equation 2-24 that, the multiplication of 95th percentile

values of three identical log-normally distributed variables yields the 99.78th percentile of the simulated output distribution, i.e.,

$$R = \frac{\prod_i F_{0.95}\left[LN\left(\mu_{\ln x_i}, \sigma_{\ln x_i}\right)\right]}{F_{0.95}\left[\prod_i \left[LN\left(\mu_{\ln X_i}, \sigma_{\ln X_i}\right)\right]\right]} \qquad \textbf{(2-25)}$$

Using conservative values in deriving the input distributions further increases this conservatism. Two other papers [Reckhow, 1994; Shevenell and Hoffman, 1993] address the need of uncertainty analysis in risk assessments and environmental decision-making. The effect of compounded conservatism is most realized at the risk management stage wherein monetary investment plays an important role. The investment to remediate a site increases in proportion to the risks determined from the risk assessment models and thus it is essential to consider all the relevant information in yielding risk numbers that are defensible and valid.

2.3.2 Probability Distributions and Monte Carlo Simulations

The popularity of Monte Carlo simulations has been so great that an entire issue of a risk assessment journal has been devoted to this subject [HERA, 1996]. This method is used to propagate a range of numbers represented by probability distribution functions. Probability distribution function (also called cumulative distribution function, CDF) and probability density function (PDF) which represent the relative likelihood of various values, *x,* for a random variable *X*, are functional and graphical characterizations of uncertainty stemming from random data. The probability distribution is characterized by

specific parameters, usually, the first and second moments which indicate the central tendency (mean) and the variance in the data. Depending on the characteristics of the distribution obtained for a variable, a number of distributions are available to fit the data, however, normal and lognormal distributions have been the most popular in exposure assessments [Roseberry and Burmaster, 1992; Ruffle *et al.*, 1994]. Some of the most common distributions are given in Table 2-7. The simplicity and very general applicability of the Monte Carlo method has encouraged analysts to use it extensively to propagate uncertainty represented by probability distributions. In this method, there is no restriction on the form of joint input distributions or on the nature of the relationship between the input and the output [Cox and Baybutt, 1981] and, it can be used to simulate any process that is influenced by random variables. The basic version of Monte Carlo simulation consists of randomly choosing the values $\left(\alpha_1^{(i)},\ldots,\alpha_n^i\right),\ i=1,\ldots.N$, (with the help of a random number generator) from various input distributions and substituting them in a function, which is an empirical expression of the risk assessment model, to obtain the corresponding outputs $Y^{(i)}$, $i = 1,...N$. The accuracy of the output estimates increases with the number of samples. Conventional sampling procedures however tend to unevenly select values that are close to the mean and hence can result in estimates that are concentrated more towards the center of the distribution (this is a big problem in cases where the distribution has long tails). To avoid this problem, an improved sampling procedure, Latin Hypercube Sampling (LHS) based on the stratification of the sample space has been developed by Iman, Davenport and Zeigler [Morgan and Henrion, 1990].

Table 2-7: Some common probability distributions and their means and variances

Type	Functional Representation	Mean (μ)	Variance (σ^2)
Uniform	PDF[a]: $\frac{1}{b-a};\ a \le x \le b$ CDF[b]: $\frac{x-a}{b-a}$ Parameters of distribution: a and b	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal	PDF: $\frac{1}{(2\pi)^{1/2}\sigma} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right);\quad -\infty \le x \le \infty$ CDF: No closed form solution. Determined from tables. Parameters of distribution: μ, σ	μ	σ
Log-Normal	PDF: $\frac{1}{\sigma x(2\pi)^{1/2}} exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right);\quad 0 \le x \le \infty$ CDF: Computed from normal distribution of ln(x). Parameters of distribution: μ, σ	$exp(\mu+\frac{\sigma^2}{2})$	$w(w-1)v;$ $w=\exp(\sigma^2)$ and $v=\exp(2\mu)$
Exponential	PDF: $\lambda exp(-\lambda x);\quad 0 \le x \le \infty$ CDF: $1-exp(-\lambda x)$ Parameters of distribution: λ	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Triangular	PDF: $\frac{2(x-a)}{(b-a)(c-a)};\quad a \le x \le c$ $\frac{2(b-x)}{(b-a)(b-c)};\quad c < x \le b$ $0\quad otherwise$ CDF: $0\quad x < a$ $\frac{(x-a)^2}{(b-a)(c-a)};\quad a \le x \le c$ $1-\frac{(b-x)^2}{(b-a)(b-c)};\quad c < x \le b$ $1\quad b < x$ Parameters of distribution: a, b (endpoints) and c is the point corresponding to the maximum on the triangle.	$\frac{a+b+c}{3}$	$\frac{m-n}{18};$ $m=a^2+b^2+c^2$ *and* $n=ab-ac-bc$

[a]Probability density function, for a continuous function, $PDF = \frac{d(CDF)}{dx}$

[b]Cumulative distribution function, $\text{CDF} = P[X \le x]$, determines the probability that the random variable takes on values less than x.

In this method, the input distributions (cumulative probability distributions) are divided into *m* equiprobable intervals. A single value is then randomly sampled from each of these *m* intervals and paired with similar samples from other input distributions. In this version of the method, as the samples are more evenly spread, a better representation of the mean, variance, and other parameters of the distribution is possible.

In recent years, Monte Carlo simulation with LHS sampling has gained much popularity among risk assessors to propagate probability distributions that represent the uncertainty in exposure concentrations. A thorough exposition of this method as applied to human health risk assessment is given by Thompson *et al.* [1992]. Smith [1994] conducted a health risk assessment of individuals exposed to volatile solvents by drinking water ingestion and inhalation using this approach. Hoffman and Hammonds [1994] demonstrate the use of the Monte Carlo approach to propagate combined uncertainty and variability (referred to as type A and type B uncertainty). In propagating such combined variables, probability distributions are provided to represent uncertainty and variability.

The main drawbacks of the Monte Carlo approach are its inability to support sensitivity analysis, high computational effort, and the dependence on the accuracy of the probability distributions. These drawbacks can sometimes be overcome by efficient sampling techniques and the inclusion of sensitivity analysis techniques into the analysis. Burmaster and Anderson [1994] lay out 14 principles of practice that can assist one in performing valid Monte Carlo risk assessments. To enhance the accuracy of input distributions, methods of selecting the appropriate distributions for all the variables has been studied in depth by Seiler and Alvarez [1996]. Other shortcomings implicit in the

use of Monte Carlo simulations are offered by Ferson [1996]. In addition to conventional Monte Carlo simulations, fuzzy arithmetic in the form of interval analysis has been added to the sampling scheme for the simulation of risk to fishery population [Ferson, 1993]. In this method, samples are taken as fuzzy numbers and then combined using interval analysis techniques (introduced in Section 2.3.3).

2.3.3 Interval Methods

Interval methods can be used to propagate uncertainty derived using all of the uncertainty quantifying concepts, however, the implementation of these methods varies from concept to concept. In interval analysis, conventional mathematical operations of sum, product and division are used to propagate intervals bounded by a minimum and a maximum value. Interval methods have been applied recently as a technique in back calculating soil cleanup targets [Frey, 1996; Burmaster *et al.*, 1995]. In this method, given three intervals, $X_1 = [x_{11}, x_{12}]$, $X_2 = [x_{21}, x_{22}]$, $X_3 = [x_{31}, x_{32}]$, wherein the first value in the interval represents a min and the other the max of the variable and, if the output variable R is expressed as,

$$R = \frac{X_1 X_2}{X_3} \qquad \textbf{(2-26)}$$

then R is given as,

$$R = \frac{[x_{11}x_{21}, x_{12}x_{22}]}{[x_{31}, x_{32}]} = \left[\frac{x_{11}x_{21}}{x_{32}}, \frac{x_{12}x_{22}}{x_{31}}\right];\ \forall x_{ij} > 0 \qquad \textbf{(2-27)}$$

Interval analysis methods are also used for propagating uncertainty characterized as membership functions derived from fuzzy set theory. In this application, intervals are formed by obtaining λ-cut levels for each input whose inner and outer values are then propagated using the following set of rules [Ross, 1995]:

$$
\begin{aligned}
&X_1 + X_2 = \left[x_{11} + x_{21}, x_{12} + x_{22}\right] \\
&X_1 - X_2 = \left[x_{11} - x_{22}, x_{12} - x_{21}\right] \\
&X_1 X_2 \quad = \left[\min(x_{11}x_{21}, x_{11}x_{22}, x_{12}x_{21}, x_{12}x_{22}), \max(x_{11}x_{21}, x_{11}x_{22}, x_{12}x_{21}, x_{12}x_{22})\right] \quad \textbf{(2-28)} \\
&X_1 \div X_2 = \left[x_{11}, x_{12}\right] \cdot \left[\frac{1}{x_{22}}, \frac{1}{x_{21}}\right] \quad \text{provided that } \left[x_{21}, x_{22}\right] \notin 0
\end{aligned}
$$

The resulting output intervals are subsequently plotted at various λ-cuts to obtain the output membership function. Other variations of interval mathematics based on λ-cuts have been developed [Dong, Shaw and Wong, 1985].

In addition to the interval analysis used in combination with the Monte Carlo simulation [Ferson, 1993] as presented in Section 2.3.2, Ferson and Kuhn [1992] also introduce interval analysis in conjunction with fuzzy arithmetic in propagating uncertainty in ecological risk assessment. In a more recent paper, Ferson and Ginzburg [1996] argue that interval analysis as opposed to probability theory is a more appropriate method for propagating uncertainty resulting from partial ignorance (systematic measurement errors or subjective uncertainty).

2.3.4 Linguistic Approach (No relation to fuzzy set theory)

Another method of uncertainty characterization as developed by the Department of Defense [Department of Defense, 1994], accounts for uncertainty by assigning

linguistic terms, such as, high, medium and low to the variables that are used to rank the hazardous waste sites. No formal uncertainty analysis is carried out per se. This method deviates from the conventional approach in allowing a comparison of risks between sites rather than the calculation of absolute risks at a specific site. The risks at sites are compared subjectively using a set of linguistic labels determined from a set of linguistic terms given by three factors: contaminant hazard factor (CHF); migration pathway factor (MPF); and receptor factor (RF). While the intensity of the MPF and RPF is determined purely by subjective judgment, a more objective criteria (a ratio between the maximum concentration of a contaminant detected at the site and the standard for the contaminant) is used for CHF. Table 2-8 lists the linguistic terms and the associated subjectively defined intensity factors.

The determined linguistic variables are then propagated by using a set of pre-defined matrices to yield the finals risks (Table 2-8). The resulting risks as demonstrated in Table 2-8 are defined by linguistic terms, high (H), medium (M) and Low (L) determined according to the corresponding combination of the CHF, MPF and RF.

Table 2-8: Assignment of linguistic intensity labels to the site evaluation factors [Department of Defense, 1994]

Contaminant Hazard Factor (CHF)	
Significant	Sum of ratios [maximum concentration/ standard] > 100
Moderate	Sum of ratios [maximum concentration/ standard] = 2 – 100
Minimal	Sum of ratios [maximum concentration/ standard] < 2
Migration Pathway Factor (MPF)	
Evident	Analytical data or observable evidence indicates that contamination in the media is moving away from the source and towards the point of exposure
Potential	Possibility for contamination to be present at or migrate to a point of exposure; or information is not sufficient to make a determination of Evident or Confined
Confined	Information indicates that the potential for contaminant migration from the source is limited (due to geological structures or physical controls)
Receptor Factor (RF)	
Identified	Receptors identified as having access to the contaminated media
Potential	There is a potential for receptors to have access to the contaminated media
Limited	Little or no potential for receptors to have access to the contaminated media

2.3.5 A Short Note on Combining Uncertainty Analysis Models with Mechanistic Models

Mathematical uncertainty analysis is a useful tool that can only be used as an aid to the model parameter uncertainties and the relation between the parameters in a predefined model. It is however more crucial that uncertainties be reduced as much as they can by improving the models themselves. There have been many efforts in the recent years that have focused on improving the validity of risk assessment models by better understanding the dosimetry and biological mechanisms involved in the toxic effect of a hazardous chemical. The models that have seen most growth in the risk assessment community are the physiologically based pharmacokinetic models (PBPK) and biologically based dose response models (BBDR) [Goddard and Kewski, 1995, Leung and Paustenbach, 1995] together known as the pharmocokinetic/pharmacodynamic models (PBPK/PD). The primary purpose of PBPK models is to predict the magnitude of dose that reaches a specific organ and in turn to predict the dose-response relationship more accurately. Consisting of various compartments that represent an organ or a tissue group, they are described by a series of differential mass-balance and biotransformation equations to predict the amount of toxic moiety being absorbed, transformed and eliminated in each compartment. The differential equations are then solved numerically to compute the amount of chemical that is directly related to a specific response in the target organ.

As opposed to the PBPK models that represent the pharmacokinetic behavior of various tissues, BBDR models are pharmacodynamic models that model the effect of a

toxicant on the human systems. To date, the most widely applied BBDR model has been the simplified two stage clonal expansion model for carcinogenesis, also known as the MVK model [Moolgavkar and Venzon, 1979, Moolgavkar and Knudson, 1981]. The main constructs of this model are defining and modeling two stages of initiation and promotion, that are crucial in the transformation of a normal cell into a malignant cell. The model assumes that a pool of normal cells *X(s)* are transformed into an intermediate cell at the rate of $\nu(s)$. The intermediate cells further divide into two intermediate cells at the rate of $\alpha(s)$ and precisely into an intermediate and a malignant cell at the rate of $\mu(s)$. An approximation of the hazard function *h(t)* derived from the MVK model for low tumor incidence is given by [Moolgavkar and Luebeck, 1990]:

$$h(t) \approx \mu(t)\int_0^t \left\{ \nu(s)X(s)\exp\int_s^t [\alpha(u)-\beta(u)]du \right\} ds \qquad \textbf{(2-29)}$$

The dissection of the above mechanistic models reveals that though they are superior to macro-statistical models in representing the risks more accurately, they are more prone to numerical uncertainties as the number of input parameters significantly increases. For example, in the PBPK models, there is uncertainty in the estimation of the physiological, biochemical and physico-chemical parameters [Farrar *et al*., 1989] and, the two-stage carcinogenesis model is inherently stochastic. Even in the simplest case wherein the hazard function is approximated by a deterministic solution as given in Equation 2-29, the two-stage model consists of parameters such as cell division (α), cell death (β) and mutation rates (μ) that are uncertain. The parallel developments in mechanistic modeling and uncertainty analysis have resulted in some combination of

uncertainty analysis and mechanistic modeling, especially in a few PBPK models [Woodruff *et al*., 1992, Cronin *et al*., 1995, Thomas *et al*., 1996]. However, uncertainty analysis models have been limited to probabilistic methods and Monte Carlo analysis. Cronin *et al*. use normal, dirichlet and lognormal distributions to replace mean values that are commonly used to represent parameters such as flow rates, partition coefficients and metabolic constants and use Monte Carlo analysis to propagate these distributions through a PBPK model [Cronin *et al*., 1995]. Similarly, Thomas *et al*. used a PBPK model with Monte Carlo simulation to study the effect of interindividual variability in physiological parameters on the biological exposure indices (BEIs) for benzene, carbon tetrachloride, chloroform, methylene chloride, and trichloroethylene [Thomas *et al*., 1996]. As biological models grow more in complexity and application, there is a greater need for an in-depth formal uncertainty analysis, as the success and validity of a model eventually rests on how well the available information is used and represented.

2.4 Conclusions

Until recently, uncertainties in risk assessments were characterized qualitatively, and not quantitatively, with the qualitative aspect being replaced by overly conservative estimates. EPA guidelines recommended the use of 95th percentile values for calculating exposures and safety factors in the orders of magnitude for the calculation of reference doses. These methods rely on point estimates and fail to capture the entire range of uncertainty. It has been shown that the reliance on such measures usually leads to an overestimation of the risk values to unrealistic levels, and provides one with a false sense

of security of the values [Burmaster 1993; Bogen, 1994]. Though the methods can be justified to be protective of sensitive sub-populations, uncertainty is not completely represented. Research in the quantitative characterization of uncertainties in human health risk assessment is still in its' infancy, with most advances having taken place in the last decade or so. Even though many papers have been written in this field, most deal with the propagation of numerical values of uncertainties derived from probabilistic concepts. Monte Carlo simulation has been the most widely used method for propagating uncertainties. Though Monte Carlo analysis has proven to be a powerful tool in propagating uncertainties, one needs to exercise caution in using this technique as the method is dependent on the choice of the distributions for the input parameters and the dependency structure of the model. In a number of cases the analyst has to rely on scarce data to derive precise distributions. In cases where data is limited it seems more appropriate to model the uncertainties using methods that are based on weaker axioms. It should also be noted that various correlations among the model parameters may exist and Monte Carlo analysis conducted without a good knowledge of the dependencies between parameters can yield erroneous outputs [Ferson, 1996 and Frederick, 1995]. There are, however, alternative methods such as fuzzy set theory, evidence theory and possibility theory that have proven to be useful in modeling some types of uncertainty. The lack of maturity of some of these alternative methods has hindered the scientific community from embracing newer concepts. In the next chapter of this dissertation we present methods that expedite the application of possibility theory to risk assessments. These new methods are useful in deriving possibility distributions from empirical data and are meant to assist an analyst in representing parametric uncertainties.

Chapter 3 Theoretical Development of Possibility Distributions

The purpose of this chapter is to introduce methods to derive possibility distributions for representing sparse or conflicting data.

In cases where there is a severe lack of information it is natural for analysts to quantify ambiguous and vague statements as a range of numbers. The less the supporting evidence is the more imprecise and vague the statements usually are. To give precise statements about the values of variables would be to assume more information than is available. Though probability theory can be used to model randomness, it fails to consider other types of uncertainty, further it is extremely difficult to model these other types as random variables. In the absence of complete data, variations in the true value of the parameters of a probability distribution have usually been modeled using second-order distributions where some other distribution models the uncertainty in the parameters. The parameters are, however, not necessarily random or even if they are, it can be computationally expensive to model these parameters as such. It should be noted

that in most situations it is not the chance that a particular value occurs that is of interest, but the actual variation in the variable itself. Thus it might be prudent to use models that are not as constrained as probabilistic models. Also, in many cases representative power (depicting the current state of the uncertainty) of the model might be more useful than the predictive power (predicting the future state). Due to the shortcomings of the current methods to model all the uncertainties, new methods are required to model uncertainty that is not always random. Possibility theory offers a flexible and a robust framework to derive and naturally represent uncertainty from data that is vague or ambiguous. It is especially useful during the explorative phases of risk assessments where the available data do not satisfy the statistical significance requirements, or the heterogeneity of the data demand extensive computations for the task at hand. To date, due to its novelty and lack of adequate methods to generate possibility distributions, possibility theory has had limited applications in risk assessments. Though significant progress has been made in the derivation of possibility distributions for representing expert judgments [Sandri *et al.*, 1995; Walley, 1991; Walley and de Cooman, 2001], methods for deriving possibility distributions from empirical data have been rare and few [Joslyn, 1994; Joslyn 1997a; Joslyn, 1997b]. In this chapter we provide alternate methods for developing possibility distributions from various types of empirical data.

3.1 Possibilistic Elicitation and Representation

The challenge of any uncertainty analyst is to determine the appropriate method to calculate and represent the uncertainties of the model parameters. The discussion in

Chapter 2 outlined some of these methods, however, most of the methods were restricted to probability theory wherein probabilities are non-overlapping, disjoint sets of numbers with well-defined boundaries. In the case of possibility theory, the form of the representation scheme is not as obvious as those derived for probabilistic analysis. Literature on derivation of possibility distributions is very limited [Klir, 1997], especially in the derivation of empirical possibility distributions [Joslyn, 1994]. Possibility theory has long been associated with fuzzy set theory [Zadeh, 1978], in that possibility distributions were considered to be membership functions. If $\mu_A(x)$ represents the membership of a fuzzy set defined over a universe X, then the possibility distribution is a fuzzy set if,

$$\pi_A(x) = \mu_A(x) \qquad \textbf{(3-1)}$$

The association of possibility theory with fuzzy set theory has led to the development of possibility theory as a model for subjective judgments and hence possibility distributions were derived for modeling linguistic variables (such as in fuzzy set theory). This however has led to constructing possibility measures that satisfy some linguistic criteria. In particular, fuzzy sets represented by membership functions model vagueness by specifying the degree to which an element x belongs to the set A, while possibility distributions represent gradations of possibility of an element belonging to consonant intervals A_i.

Another approach to the development of possibility distribution functions is through the transformation of probability distribution functions [Klir, 1992, 1993; Dubois *et al*., 1993]. Given a unimodal probability density function *p* with mode x_o, a possibility

distribution can be derived by the following transformation:

$$\pi(x) = \pi(x') = \int_{x_l}^{x} p(v)dv + \int_{x'}^{x_u} p(v)dv \qquad \textbf{(3-2)}$$

where x' is such that $p(x) = p(x') < p(x_o)$, and, x_l, x_u are the lower and upper limits of the probability distribution.

The above two methods for computing possibility distributions do not completely assist modeling empirical interval data as the fuzzy set approach, on the basis of its interpretation, has been predominantly used to model subjective data while the probability-possibility transformation is based on randomness only.

Possibility distributions can also be viewed as resulting from consonant random sets. This concept arises from Dempster and Shafer's theory (DST) when the evidence focuses on consonant support functions [Shafer, 1976]. In the case of deriving empirical possibility distributions it is more natural to consider random sets (or intervals) and build a distribution based on the original set of interval data. This interpretation is more realistic as experimental observations are usually recorded as ranges of number. To this effect, Joslyn [1992, 1997b] has developed a method that calculates a possibility histogram from random sets. Joslyn's method, however, derives possibility measures based on only consistent random sets (where measurements are not required to be nested intervals, but it is only sufficient that all the measurements intersect) rather than consonant sets (measurements are represented as nested intervals). Though possibility distributions can be derived from measurements that satisfy the consistency criteria only, we enforce the property of consonance to aid in the development of possibility

distributions. There is no conclusive evidence on which assumption leads to the most appropriate possibility distribution, however, assumption of consonance in this dissertation has allowed more natural derivation of possibility distributions from a variety of measurement types.

Methods alternate to Joslyn's method are proposed to build consonant possibility distributions from not only consistent measurements but also other kinds of measurement types. New methods are developed for consistent as well as non-consistent and disjoint measurements. In the selection of intervals and calculation of weights from any of these measurements, the underlying evidence is considered to be coherent i.e., evidence points in one direction and hence is consonant. Consonance is ensured by appropriate selection of overlapping intervals and assignment of weights derived by conventional frequency analysis. Assumption of consonance is one of the major assumptions in possibility theory. This assumption seems to be reasonable given that there is only one or a few values out of many that are possible and that inductive reasoning suggests that an analysts' result should converge to this one value. This does not mandate that this value is certain to occur but that the evidence suggests that this is the value with the highest possibility. Physically this indicates that if an event that is a set of values does not include the most possible value(s) then it is not necessary for the event to occur but it is possible to occur. This is consistent with the fact that something has to be completely possible before it is necessary. The requirement of consonance also ensures that possibility distributions are either strictly decreasing or strictly increasing monotonic functions or unimodal non-monotonic distributions such that, given the most possible interval q, and for any value x, the distribution $\pi(x)$ is strictly increasing when $x \leq q$

and strictly decreasing when $x \geq q$.

In addition to the requirements presented above, other important definitions that formalize the basic concepts required for the development of possibility distributions are presented formally in the sub-sections below.

Definition 3-1: Random set

Given a function m on a power set $\wp(\Omega)$, such that $m : \wp(\Omega) \rightarrow [0,1]$, a random set S_R, is defined as a vector of tuples, $\{\langle A_j, m_j \rangle\}$ where $A_j \in \wp(\Omega)$, $1 \leq j \leq N$, $1 \leq N \leq 2^n$ and $m_j = m(A_j)$, n is the number of elements in Ω, N is the number of elements in the random set, and m_j can be interpreted as the weights attached to each component A_j.

Definition 3-2: Focal set

A set of all A_j for which $m_j > 0$ forms the focal set F,

$$F = \{A_j \in \wp(\Omega) \mid m_j > 0\}$$

Definition 3-3: Random interval

A random set is a random interval if it is defined on a real line $\Re$, i.e., if the universe of discourse, $\Omega = \Re$. When a random interval is used, individual elements (or numbers) will be represented by $x_k, k = 1,2,...,n$.

Definition 3-4: Consistent random sets

A random set S_R is said to be consistent if the core of the random set is non-empty. The core of the random set is,

$$C(\mathrm{S_R}) = \bigcap_{A_j \in F} A_j \qquad \textbf{(3-3)}$$

For a consistent random set,

$$C(\mathrm{S_R}) \neq \varnothing \qquad \textbf{(3-4)}$$

Definition 3-5: Consonant random sets

A random set $\mathrm{S_R}$ is consonant, if for some permutation of the $A_j \in F$, the elements A_j of the random set are such that, $A_1 \subseteq A_2 \subseteq .... \subseteq A_N$, $1 \leq j \leq N$.

Definition 3-6: Fuzzy Measure and Possibility measure

A fuzzy measure $\upsilon : \wp(\Omega) \rightarrow [0,1]$ is required to satisfy the following axioms:

(1) $\upsilon(\varnothing) = 0$ and $\upsilon(\Omega) = 1$

(2) $\forall \mathrm{A}, \mathrm{B} \subseteq \Omega$, if $\mathrm{A} \subseteq \mathrm{B}$, $\upsilon(\mathrm{A}) \leq \upsilon(\mathrm{B})$

Now, a fuzzy measure is a possibility measure $\Pi : \wp(\Omega) \rightarrow [0,1]$ if it satisfies the following property in addition to the ones listed above:

(3) For a collection of intervals A_j, $1 \leq j \leq N$, the possibility of the union of finite number of events,

$$\Pi(\bigcup_{j=1}^{J} A_j) = \max_{j \in J} \Pi(A_j), \ \forall A_j, \ j \in J \qquad \textbf{(3-5)}$$

Possibility measures are plausibility measures as derived by Dempster-Shafer theory for those that are defined over consonant sets.

Definition 3-7: Necessity measure

The dual of a possibility measure is a necessity measure $N : \wp(\Omega) \to [0,1]$, and is given as,

$$N(A) = 1 - \Pi(\overline{A}) \quad \textbf{(3-6)}$$

Necessity measures are belief measures as derived by Dempster-Shafer theory for those that are defined over consonant sets.

Definition 3-8: Possibility distribution function

Every possibility measure is uniquely associated with a possibility distribution function, $\pi : \Omega \to [0,1]$, defined by $\pi(\omega) = \Pi(\{\omega\}), \omega \in \Omega$.

From the above it can then be derived that,

$$\forall A \in \wp(\Omega), \Pi(A) = \max\left(\pi(\omega) \middle| \omega \in A\right) \quad \textbf{(3-7)}$$

In the case when the distribution is continuous, max is replaced by sup.

From Equation 3-6, a necessity measure is given in terms of the possibility distribution as,

$$N(A) = \min(1 - \pi(\omega) \middle| \omega \notin A) \quad \textbf{(3-8)}$$

In the case when the distribution is continuous, min is replaced by inf.

Definition 3-9: **Possibility measures are super-additive and necessity measures are sub-additive**

Given a possibility and necessity measure on sets A and B,

$$\Pi(A) + \Pi(\overline{A}) \geq 1 \quad \textbf{(3-9)}$$

$$N(A) + N(\overline{A}) \leq 1 \quad \textbf{(3-10)}$$

Definition 3-10: Possibilistic weights

Given a set of consonant intervals on $\Re$, $A_1 \subseteq A_2 \subseteq .... \subseteq A_N$, $1 \leq j \leq N$, if $m(A_k)$ represents the weights assigned to the k th consonant interval and if the elements of the intervals are represented by x_k, and can be arranged such that, $\pi(x_{k-1}) \geq \pi(x_k) \geq \pi(x_{k+1})$ and if $A_j = \{x_1, x_2, ..., x_j\}$ then the possibility,

$$\pi(x_k) = \sum_{j=k}^{N} m(A_j) \quad \textbf{(3-11)}$$

Or alternatively, weight assigned to each interval can be obtained from the possibilities as,

$$m(A_j) = \pi(x_j) - \pi(x_{j-1}) \quad \textbf{(3-12)}$$

When deriving set-based functions, however, the weights are over sets rather than individual elements and cannot be arranged as above. Therefore, when the possibility distribution is derived from discrete random measurements represented as intervals A_j, possibilistic weights are given as,

$$\pi(x) = \sum_{x \in A_j} m(A_j) \quad \textbf{(3-13)}$$

where, $\pi(x)$ is the possibilistic weight of the elements included in the random interval A_j.

Definition 3-11: Triangular norm (T-norm)

A triangular norm (T-norm) is a binary conjunctive operator, $T:[0,1]\times[0,1]\to[0,1]$, that satisfies the following conditions,

$\forall x, y, z \in [0,1]$,

1. $T(x,y) = T(y,x)$ (**commutativity**)

2. $T(x,y) \le T(x,z)$, if $y \le z$ (**monotonicity**)

3. $T(x,T(y,z)) = T(T(x,y),z)$ (**associativity**)

4. $T(x,1) = x$

Examples of common T-norms include the product norm used in probabilistic analysis to determine the intersection of two crisp sets and the min norm used extensively in fuzzy set theory to determine an intersection of fuzzy sets (see Gupta and Qi, 1991 for a list of triangular norms).

3.2 Developing Possibility Distributions

In a typical experiment, measurements can be any of the following types of measurements: 1) consonant measurements; 2) consistent measurements; 3) non-consistent, non-disjoint measurements, 4) point estimates; and 5) disjoint measurements. The derivation of possibility distributions is specific to the form of measurement and hence the methodology used to develop the distributions should be appropriately developed. The following sections describe methods that can be used to derive

possibility distributions for each of the measurement types.

3.2.1 Developing Possibility Distributions from Consonant Measurements

Since possibility distributions are functions on consonant intervals, a possibility distribution from consonant intervals is trivially obtained and no specific methodology is developed. Simply, if $A_j, \mathrm{j} = 1.....n$ is a set of n consonant observations, then a possibility distribution is obtained as a contour function over a possibility measure as given in Definition 3-10 as,

$$\pi(x) = \sum_{x \in A_j} m(A_j) \qquad \textbf{(3-14)}$$

where, $m(A_j)$ are the weights strictly assigned to the j th observation.

3.2.2 Method I: Developing Possibility Distributions from Consistent or Non-Consistent, Non-disjoint Measurements

Consonant measurements are more easily obtained for expert judgments where the measurements can be elicited for nested data sets, however, in experimental analysis consonance at the observation level cannot be guaranteed and hence consistent measurements as defined in Definition 3-4 are somewhat more common than consonant distributions. Just as consistent measurements are more common than consonant measurements, partially overlapping non-consistent measurements are even more common. A method for deriving consonant intervals from such non-consonant measurements is derived in this section. The method presented here differs from that

presented earlier in the literature [Joslyn, 1994] in that the measurements are not required to be consistent and a possibility distribution is defined over consonant intervals that are derived from the original consistent or non-consistent measurements. The requirement for using this method is however that the measurements are non-disjoint (a method for deriving possibility distributions for disjoint measurements in given in subsequent sections). Transforming non-consistent, non-disjoint intervals to consonant intervals allows one to define possibility distributions as a set-based rather than a point function. Noting that possibility distributions are strictly monotonic or non-monotonic unimodal functions on consonant intervals a methodology that derives consonance through intersecting elements of the measurements and then redistributes the weights from the non-consonant to consonant intervals is presented in the following sub-sections.

3.2.2.1 Deriving Nested Intervals using Method of Intersecting Sets

Consider a system that can be described within a domain $X = \{x_1, x_2, ..., x_n\}$, the behavior of which is described by evidence obtained as observations over a collection of sets, F. Let $F = \left\{\left\langle A_j, w_j \right\rangle\right\}$ represent the original set of focal elements along with their weights. If M measurements are observed, then the weights w_i of each observation A_j is calculated by frequency analysis as,

$$w_j = \frac{\mathrm{n}(A_j)}{\mathrm{M}} \qquad \textbf{(3-15)}$$

where, $n(A_j)$ is the frequency count of set A_j, and $\sum_j w_j = 1$, $\sum_{\mathrm{j}} n(A_j) = \mathrm{M}$.

In conventional probability analysis, the observations A_1 and A_2 are disjoint such that $A_1 \cap A_2 = \varnothing$, and the frequency of occurrence of any event A_j is simply the ratio of the count of a particular event and the total number of occurrences of all the events. Such disjoint measurements are, however, rare in the real world; instead, measurements are usually such that F consists of overlapping sets A_j. In a measuring device that reveals evidence that is coherent, the sets A_j are nested, in which case possibility distributions are trivially obtained over a set of consonant intervals. If F, however, consists of non-consonant sets, the vector of original focal elements F has to be transformed such that consonance is achieved. In deriving the set of consonant focal elements it is assumed that the underlying evidence is coherent such that the observations obtained should reveal intervals that decrease in specificity. This is accomplished such that the smallest focal element (having the least specificity) is selected as the interval that has the most intersections with the observations and the next smallest one as that having second most intervals and henceforth. This process yields Q unique countable intersections and a set $G = \{\langle B_k, \nu(B_k)\rangle\}$, $k = 1,2,...,Q$ of intersections, where B_k is the interval obtained by the k th intersection of original intervals A_i and A_j, and $\nu(B_k)$ is the weight assigned to the corresponding B_k. Union of all the sets, $S = \bigcup_{A_j \in F} A_j$ where $w(A_j) \neq 0$ forms the support interval and is derived as:

$$S = \left[\min_{A_i \in F, x \in A_i} (x), \max_{A_i \in F, x \in A_i} (x) \right] \qquad \textbf{(3-16)}$$

Weights ν_k assigned to each of the Q elements of G are determined by utilizing

any of the triangular norms (see Definition 3-11) for conjunctions as $\nu(\mathrm{B_k}) = \mathrm{T}(A_i, A_j)_k$; where $\mathrm{T}(A_i, A_j)_k$ is any conjunctive triangular norm operating on the sets A_i and A_j that form the k th intersection. Since the dependencies between various sets is unknown and it is judicious to choose the most conservative estimate, the min norm is considered to be most appropriate to determine weights $\nu(\mathrm{B_k})$ of the conjunction of sets A_i and A_j, and is given as,

$$\nu(\mathrm{B_k}) = \min\left(w_i, w_j\right)_k \qquad \textbf{(3-17)}$$

Since a min norm is based on a weaker axiom of non-additivity, the weights do not necessarily add to one, and hence the weights derived are normalized as,

$$\eta(\mathrm{B_k}) = \frac{\nu(\mathrm{B_k})}{\sum_{\mathrm{k=1}}^{\mathrm{Q}} \nu(\mathrm{B_k})} \qquad \textbf{(3-18)}$$

such that, $\sum_{\mathrm{k}} \eta(\mathrm{B_k}) = 1.$

In deriving the elements of G, however, some elements by the virtue of their origin from the intersections from parent sets (sets from which the intervals were derived) tend to be consonant with the parent set while they are not necessarily consonant with other parent sets. Therefore, if Q represents the total number of focal elements in the focal set G derived from the intersections of original measurements, there are $\mathrm{Q_H}$ elements in the consonant set H, and $\mathrm{Q_I}$ elements in the non-consonant set I, such that,

$$\sum_{\mathrm{B_k} \in \mathrm{Q_H}} \eta(\mathrm{B_k}) + \sum_{\mathrm{B_k} \in \mathrm{Q_I}} \eta(\mathrm{B_k}) = 1.0 \qquad \textbf{(3-19)}$$

Consonant intervals can be extracted from the intersection set G as a combination of frequency analysis and expert judgments. The most possible interval being one that occurs most frequently and that in which an expert has the most confidence in. Once the task of selecting consonant intervals is accomplished, the weight from the remaining non-consonant intervals is redistributed to the consonant intervals in such a manner that the total information from underlying evidence is preserved.

3.2.2.2 Redistribution Factor

Weights are redistributed from non-consonant to consonant sets according to the dissonance between individual intersecting sets (the conflict between two sets). The logic used here is that the higher the similarity between two sets (lower the conflict) the greater is the weight that can be transferred between the two sets. Parameters that are useful in the redistribution of weight are identified as cardinality - $\left|H_i\right|$, and the number of common elements between the sets - $\left|H_i \cap I_j\right|$. In case of non-integer intervals, cardinality $\left|.\right|$ can be replaced by the length l of the interval defined over a real line and the set of real numbers comprising the intersecting interval can be determined as the length common to both the intervals. If β represents the similarity of two sets given as,

$$\beta_{ij} = \frac{\left|H_i \cap I_j\right|}{\left|H_i\right|} \qquad \textbf{(3-20)}$$

or for a set of real numbers,

$$\beta_{ij} = \frac{l\left[\min \omega_i \middle| \omega_i \in H_i \cap I_j \neq \varnothing, \max \omega_i \middle| \omega_i \in H_i \cap I_j \neq \varnothing\right]}{l[H_i]} \quad \textbf{(3-21)}$$

where,

$l[.]$ denotes the length of the interval and is simply given as $l[a,b] = b - a$,

then, a redistribution factor κ is computed as,

$$\kappa_{ij} = \frac{\beta_{ij}}{\sum_{i=1}^{Q_H} \beta_{ij}} \quad \textbf{(3-22)}$$

such that, for any j, $\sum_{i=1}^{Q_H} \kappa_{ij} = 1.0$.

Redistribution factor can be viewed as the fraction of the weight that is transferred from the non-consonant to the consonant interval. The redistribution weight ρ is then calculated by determining the weight of the non-consonant interval I_j that is transferred to the corresponding consonant interval H_i. Therefore, for the *i* th consonant interval and *j* th non-consonant interval,

$$\rho_{ij} = \kappa_{ij} * \eta(I_j) \quad \textbf{(3-23)}$$

From Equation 3-21 it is clear that when I_j intersects with H_i and no other set then β is 1 thus assigning the entire weight of I to H, and also, when H does not intersect with any portion of I, β is 0. The final weights of the initial consonant sets as determined after the redistribution is given over the entire set of non-consonant intervals as,

$$m(\mathrm{H_i}) = \eta(\mathrm{H_i}) + \sum_{j=1}^{Q_I} \rho_{ij} = \eta(\mathrm{H_i}) + \sum_{j=1}^{Q_I} \kappa_{ij}\eta(\mathrm{I_j}) \qquad \textbf{(3-24)}$$

The above process entails that the total weight is preserved among the consonant data intervals. It can be proved that $\sum_{i=1}^{Q_H} m(\mathrm{H_i}) = 1.0$. From Equation 3-24 and with the constraint $\sum_{i=1}^{Q_H} \kappa_{ij} = 1.0$ from the previous page, we have

$$\begin{aligned}
\sum_{i=1}^{Q_H} \mathrm{m}(\mathrm{H_i}) &= \sum_{i=1}^{Q_H} \eta(\mathrm{H_i}) + \sum_{i=1}^{Q_H}\sum_{j=1}^{Q_I} \kappa_{ij}\eta(\mathrm{I_j}) \\
&= \sum_{i=1}^{Q_H} \eta(\mathrm{H_i}) + \sum_{j=1}^{Q_I} \left(\eta(\mathrm{I_j}) \sum_{i=1}^{Q_H} \kappa_{ij} \right) \\
&= \sum_{i=1}^{Q_H} \eta(\mathrm{H_i}) + \sum_{j=1}^{Q_I} \eta(\mathrm{I_j}) \\
&= 1.0
\end{aligned}$$

The possibility distribution from the weights is then obtained as follows:

$$\pi(x) = \sum_{x \in \mathrm{H_i}} m(\mathrm{H_i}) \qquad \textbf{(3-25)}$$

3.2.2.3 Algorithm

An outline of the algorithm for deriving a possibility distribution from non-consistent and non-disjoint intervals is given below:

Step 1: Find the intersections of the various intervals and rank the intervals according to their cardinality, $\left|\mathrm{A_i} \cap \mathrm{A_j}\right|$.

Step 2: Calculate the weights of intervals of Step 1 by applying t-norms to the intersecting intervals (the min method is recommended when the dependency

between the intervals is unknown), $\nu(\mathrm{B_k}) = \min(w_i, w_j)_k$.

Step 3: Choose the smallest interval with the most intersections as the reference interval and select the subsequent intervals such that they are consonant, $\mathrm{A_1} \subseteq \mathrm{A_2} \subseteq ..... \subseteq \mathrm{A_{Q\text{-}Q_1}}$.

Step 4: Redistribute the weights from the non-consonant to the consonant interval by the method outlined above.

3.2.2.4 Example:

In this example we show the application of the above steps and compare the method with that of Joslyn's. Consider data obtained as non-consonant intervals (given in Table 3-1, Figure 3-1), where the imprecision in the data is assumed to be due to the lack of the instrument to record precise values or due to the observer's inability to record precise numbers. Data such as these are common in the real world wherein they are presented as a number given within a certain error value. In Table 3-1, 4 sets of such observations have been recorded. From the graphical representation as shown in Figure 3-1, it can be seen that the set of numbers within the range [4,5] intersect the observation sets the most (4 times).

Table 3-1: Consistent interval observations

Observation No.	Set
1	[1,4]
2	[2,5]
3	[1,5]
4	[3,6]

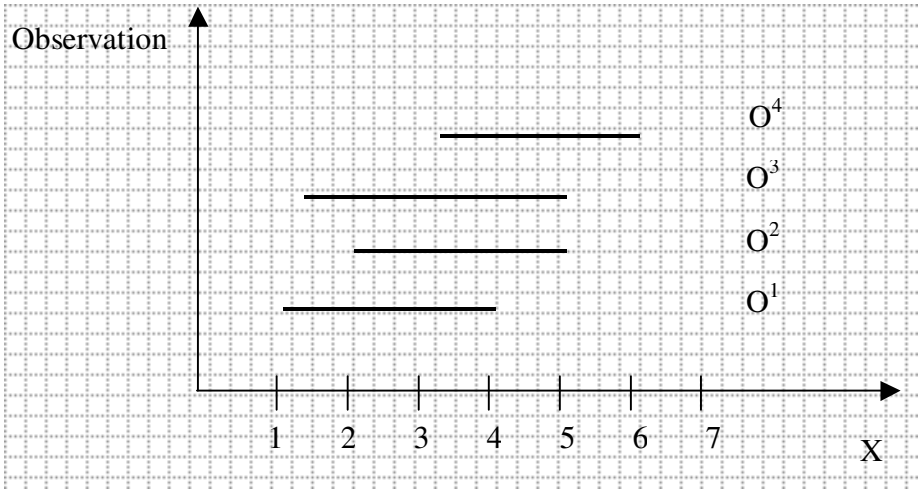


Figure 3-1: Graphical representation of observed consistent data.

Joslyn's Method

The method consists of determining the focal elements (non-recurring intervals) and computing the frequencies of each focal element's occurrence. In this example, the focal element set is given as,

$$F^E = \langle [1,4],[2,5],[1,5],[3,6] \rangle,$$

and the frequencies of occurrence are,

$$S^E = \langle \{[1,4],0.25\},\{[2,5],0.25\},\{[1,5],0.25\},\{[3,6],0.25\} \rangle$$

The possibility distribution function is now derived as, $\pi(\omega) = \sum_{\omega \in E_i} m(E_i)$

$$\pi(\omega) = \begin{cases} 0.5 & \omega \in [1,2] \\ 0.75 & \omega \in [2,3] \\ 1.0 & \omega \in [3,4] \\ 0.75 & \omega \in [4,5] \\ 0.25 & \omega \in [5,6] \end{cases}$$

The above derivation is plotted in Figure 3-2.

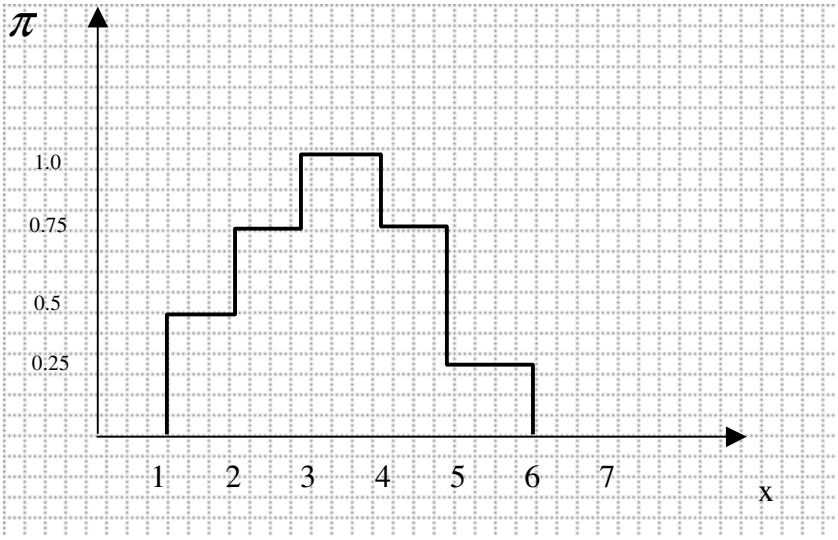


Figure 3-2: Possibility histogram derived from consistent data observations (Joslyn's Method).

Method I:

This method consists of determining the intersections of various intervals and then computing the frequency of occurrence of each of these intervals. The matrix below shows these intersecting intervals.

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} [1,4] & [2,4] & [1,4] & [3,4] \\ & [2,5] & [2,5] & [3,5] \\ & & [1,5] & [3,5] \\ & & & [3,6] \end{bmatrix} \end{array}$$

Figure 3-3: **Matrix of sets resulting from intersection of original consistent observations.**

Once the intervals are determined the frequency of occurrence of these intervals is computed. The interval forming the support of all the focal elements is calculated by taking the min and max of all observations. Table 3-2 and Table 3-3 (support, S = [1,6])

show the weights determined for each set from the original observations and those computed from the intersecting sets shown in Figure 3-3.

Table 3-2: Weights assigned to each interval according to frequency analysis

Set	Count	Weight (w_i)
[1,4]	1	0.25
[2,5]	1	0.25
[1,5]	1	0.25
[3,6]	1	0.25

Table 3-3: Weights computed for each intersecting set using the *min* norm

Set	$\nu(B_k)$	Normalized Weights ($\eta(B_k)$)
[1,4]	0.25	0.125
[2,4]	0.25	0.125
[3,4]	0.25	0.125
[2,5]	0.25	0.125
[3,5]	0.25	0.125
[1,5]	0.25	0.125
[3,6]	0.25	0.125
[1,6]	0.25	0.125

A study of the Table 3-3 (showing intersecting sets) reveals more than one consonant group of intervals. The choice of which group to choose depends on the analyst. For example, one set of consonant intervals could be, [3,4], [2,4], [2,5], [1,5], and [1,6]. This excludes three intervals, [1,4], [3,5], and [3,6]. In determining the extent of the effect of these intervals we re-distribute the weights according to the number of intersecting elements with each of the consonant intervals. For example in the case of the overlap of interval [2,5] with the first consonant interval [3,4], the number of overlapping elements (considering a granularity of 1) is 2. This value is divided by the cardinality of the

consonant interval to account for the effect of spread of each interval (the more the spread the less the effect of the weight of the non-consonant interval) (Equation 3-20). In the case of the consonant interval [3,4], the cardinality is 2, hence the weight to be distributed is factored by 1. The factored weight is then normalized over the weights of all the consonant intervals (Equation 3-22). The original weight of the consonant interval [3,4] is then added to the factored weight of the non-consonant interval (Equation 3-23 and Equation 3-24). The process of re-distribution is then repeated for other consonant intervals. By this process the weights of the consonant intervals are updated (Table 3-4) and the possibility levels are determined as given by Equation 3-25. The choice of consonant intervals in this example has been more arbitrary and has depended on expert judgments, however, more formal quantitative techniques can be used to determine the most appropriate set. Recently there have been attempts to apply the principles of entropy and similarity measures to the method in order to determine the most appropriate set of consonant intervals [Chavez, 2002].

Determining the redistribution weights:

$$k_{11} = \frac{\beta_{11}}{\beta_{11} + \beta_{21} + \beta_{31} + \beta_{41} + \beta_{51}} = \frac{1}{4.21} = 0.238;$$

Similarly,

$$k_{12} = 0.258; k_{13} = 0.272; k_{21} = 0.258; k_{22} = 0.188; k_{23} = 0.18; k_{31} = 0.178; k_{32} = 0.214; k_{33} = 0.204$$
$$k_{41} = 0.19; k_{42} = 0.171; k_{43} = 0.163; k_{51} = 0.157; k_{52} = 0.142; k_{53} = 0.18$$

The redistribution weights ρ are calculated from Equation 3-23 as,

$$\rho_{11} = \kappa_{11} * \eta(\mathrm{I}_1) = 0.238 * 0.125 = 0.02975$$

Table 3-4: Determining the similarity between focal elements for redistributing weights from non-consonant to consonant intervals

H_i	I_i	β_{ij}
[3,4]	[1,4]	$\frac{\lvert[3,4]\cap[1,4]\rvert}{\lvert 3,4\rvert}=\frac{2}{2}=1$
	[3,5]	$\frac{\lvert[3,4]\cap[3,5]\rvert}{\lvert 3,4\rvert}=\frac{2}{2}=1$
	[3,6]	$\frac{\lvert[3,4]\cap[3,6]\rvert}{\lvert 3,4\rvert}=\frac{2}{2}=1$
[2,4]	[1,4]	$\frac{\lvert[2,4]\cap[1,4]\rvert}{\lvert 2,4\rvert}=\frac{3}{3}=1$
	[3,5]	$\frac{\lvert[2,4]\cap[3,5]\rvert}{\lvert 2,4\rvert}=\frac{2}{3}=0.66$
	[3,6]	$\frac{\lvert[2,4]\cap[3,6]\rvert}{\lvert 2,4\rvert}=\frac{2}{3}=0.66$
[2,5]	[1,4]	$\frac{\lvert[2,5]\cap[1,4]\rvert}{\lvert 2,5\rvert}=\frac{3}{4}=0.75$
	[3,5]	$\frac{\lvert[2,5]\cap[3,5]\rvert}{\lvert 2,5\rvert}=\frac{3}{4}=0.75$
	[3,6]	$\frac{\lvert[2,5]\cap[3,6]\rvert}{\lvert 2,5\rvert}=\frac{3}{4}=0.75$
[1,5]	[1,4]	$\frac{\lvert[1,5]\cap[1,4]\rvert}{\lvert 1,5\rvert}=\frac{4}{5}=0.8$
	[3,5]	$\frac{\lvert[1,5]\cap[3,5]\rvert}{\lvert 1,5\rvert}=\frac{3}{5}=0.6$
	[3,6]	$\frac{\lvert[1,5]\cap[3,6]\rvert}{\lvert 1,5\rvert}=\frac{3}{5}=0.6$
[1,6]	[1,4]	$\frac{\lvert[1,6]\cap[1,4]\rvert}{\lvert 1,6\rvert}=\frac{4}{6}=0.66$
	[3,5]	$\frac{\lvert[1,6]\cap[3,5]\rvert}{\lvert 1,6\rvert}=\frac{3}{6}=0.5$
	[3,6]	$\frac{\lvert[1,6]\cap[3,6]\rvert}{\lvert 1,6\rvert}=\frac{4}{6}=0.66$

Other values of ρ are similarly computed and are added to the each of the consonant intervals to determine the final mass of each consonant interval. Therefore, using Equation 3-24,

$$m[\mathrm{H}_1] = 0.125 + \rho_{11} + \rho_{12} + \rho_{13}$$
$$= 0.125 + (0.125)(0.238) + (0.125)(0.258) + (0.125)(0.272) = 0.224$$

Similarly,

$$m[\mathrm{H}_2] = 0.125 + (0.125)(0.238) + (0.125)(0.188) + (0.125)(0.18) = 0.201$$
$$m[\mathrm{H}_3] = 0.125 + (0.125)(0.178) + (0.125)(0.214) + (0.125)(0.204) = 0.199$$
$$m[\mathrm{H}_4] = 0.125 + (0.125)(0.19) + (0.125)(0.171) + (0.125)(0.163) = 0.191$$
$$m[\mathrm{H}_5] = 0.125 + (0.125)(0.157) + (0.125)(0.142) + (0.125)(0.18) = 0.185$$

Table 3-5: Final weights of the consonant intervals

Set	Weight	$\pi(x)$
[3,4]	0.224	1.0
[2,4]	0.201	0.776
[1,4]	0.199	0.575
[1,5]	0.191	0.376
[1,6]	0.185	0.185

The possibility distribution thus derived is shown in Figure 3-4. Dotted line shows the possibility distribution as obtained using Joslyn's method. Comparison of the two distributions shows that the distribution as derived using Method I yields a distribution that assigns higher values to the intervals that are assumed consonant. This is due to the fact that the consonant intervals dominate the non-consonant intervals in the set of all intersections, and the weight is transferred from the non-consonant to the consonant portions. This might be a desirable characteristic when one intends to let the evidence lead to the most possible value by assigning more weight to the consonant portion of the data and less weight to the non-consonant portion.

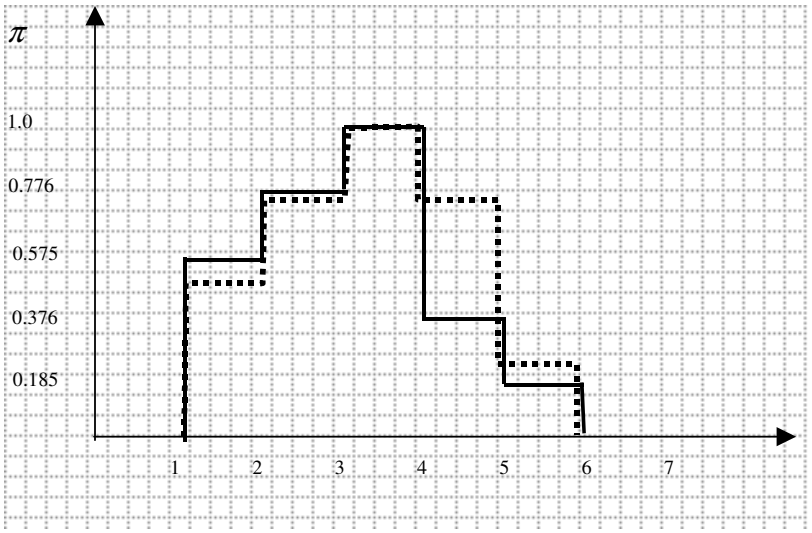


Figure 3-4: Possibility distribution as determined from the proposed Method I (Dotted line represents the distribution derived using Joslyn's method).

3.2.3 Method II: Developing Possibility Distributions from Singletons and Disjoint Measurements

When generating uncertainty distributions from point estimates or disjoint intervals, one might be tempted to use probability distribution, however, when the number of observations are too few and the parameters for deriving the distribution cannot be determined precisely it might be beneficial to use a less constrained distribution. Possibility distributions offer a viable solution to cases such as these. For example when the observations from experimental analysis or on-site measures are few and sparse as shown in Figure 3-5, finding the shape and scaling factors required for a probability distribution requires assumptions of the underlying distribution and can be a challenging task. In such situations, a possibility distribution that does not assume any randomness seems more appropriate and an approach that is consistent with possibility measures for deriving an uncertainty distribution is presented below.

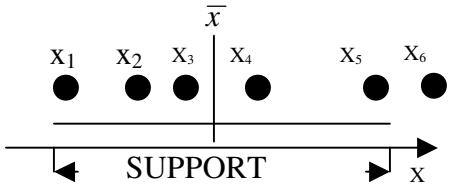


Figure 3-5: Unique point estimates in a single experiment.

3.2.3.1 Developing Possibility Distributions from Singletons

Let $x_i, i = 1....M$ be *M* observations that are points on the real line and let $w(x_i)$ be the weights assigned to each observation. Initial weights can be determined using frequency analysis as,

$$w(x_i) = \frac{M_i}{M} \qquad \textbf{(3-26)}$$

where,

M_i is the number of occurrences of the data point x_i, and

M is the total number of observations.

Given the observations and weights, standard statistical analysis techniques can be used to determine the basic properties of the data to determine a preliminary convergence point. For example the central tendency in conjunction with variance can be used to approximately determine a location for the most possible value. This seems similar to the conventional probability analysis; however, its intent differs significantly in that observations that eventually describe a possibility distribution are not assumed to represent any particular randomness in the system. The statistic used in deriving a possibility distribution, rather than indicating any functional form of the possibility

distribution merely denotes the confidence of a domain expert in a value or a range of values that are most possible and most optimal. Selection of an appropriate anchor can thus be directly based on the content of the data and/or on the expertise of the domain expert. Once the appropriate anchor is determined, a possibility distribution is determined by organizing the data points such that the most similar observations are aggregated into a cluster with the observation closest to the anchor forming the most possible interval. The concept underlying this is that the value most similar to the optimal value has a greater possibilistic weight. Since possibility distributions are assumed to be unimodal, the observations are subsequently ordered such that the observations less than the numerical value of that anchor form the increasing segment and the observations to the right form the decreasing segment of the distribution.

Therefore, if the central tendency statistic of the data, x_c, represents the anchor, and the observations are organized such that the most similar measurements are grouped together then the possibility distribution consists of an increasing segment such that, $\forall x$, $x^L{}_{i-1} \le x^L{}_i \le x^L{}_{i+1} \le x_c$, with the corresponding mappings, $\pi(x^L{}_{i-1}) \le \pi(x^L{}_i) \le \pi(x^L{}_{i+1})$, and for the decreasing segment as, $x_c \le x^U{}_{i-1} \le x^U{}_i \le x^U{}_{i+1}$ and $\pi(x^U{}_{i-1}) \ge \pi(x^U{}_i) \ge \pi(x^U{}_{i+1})$ respectively. The new ordered observations $x^L{}_i$ and $x^U{}_i$ are determined such that the most similar measurements are adjacent to each other. The support of all the singleton observations is determined as,

$$S = \left\lfloor \min_{x \in F}(x), \max_{x \in F}(x) \right\rfloor \qquad \textbf{(3-27)}$$

Nesting of intervals is then resolved such that the intervals that are most possible are

grouped together followed by those that are possible to a lesser extent and so forth. Due to their strong pattern recognition features, the concepts of hierarchical clustering algorithms [Jain, 1999] seemed most appropriate for achieving this aggregation of point estimates into nested intervals. In a hierarchical clustering algorithm, each object in a sample is compared to other objects and the most similar objects as determined by a metric, along with a cost function, is pooled together to form a cluster. For example, if $\delta(O_i)$ represents a metric that defines an arbitrary object O_i, then objects with similar δs are grouped together such that the new group $O_{New} = O_i \cup O_j$ forms the point of reference for subsequent clustering. Each new cluster O_{New} thus formed is grouped with other similar clusters iteratively until all the clusters are combined together in a hierarchy of similarities. One of the common cost functions used to achieve this grouping is based on the single-linkage clustering algorithm [Jain, 1999] and is defined as,

$$\Delta = \min_{x_i \in O_i, x_j \in O_j} (\delta(x_i, x_c), \delta(x_j, x_c)) \qquad \textbf{(3-28)}$$

where,

$$\delta(x_i, x_c) = \left(\sum_{k=1}^{d} \left(x_{i,k} - x_{c,k} \right)^p \right)^{1/p} \qquad \textbf{(3-29)}$$

When p=2, Equation 3-29 represents the well known Euclidean norm. Given that the data are given as point estimates in one-dimension (d=1), a Euclidean norm should suffice and Equation 3-29 can be re-written as,

$$\delta(x_i, x_c) = \left((x_i - x_c)^2\right)^{1/2} = |x_i - x_c| \qquad \textbf{(3-30)}$$

A dendrogram that is derived using single-linkage clustering is shown in Figure 3-6. From the dendrogram it can be seen that intervals can be nested in only one direction such that each successive interval includes the previous interval. A graphical representation of these intervals is shown in Figure 3-7. Once the observations are aggregated, intervals are determined using the end points of the newly aggregated set such that the new set,

$$C_i = \left[\min\left(x | x \in O_{New}\right), \max\left(x | x \in O_{New}\right)\right] \qquad \textbf{(3-31)}$$

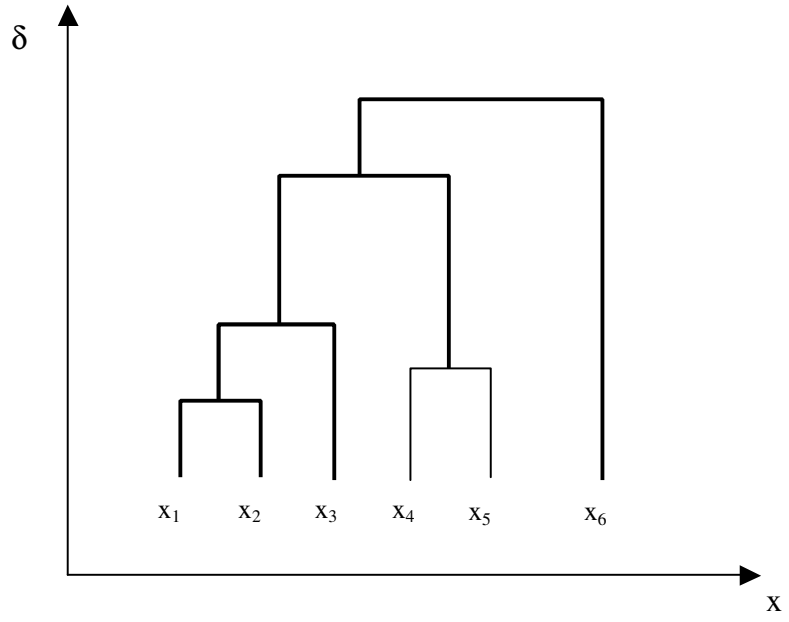


Figure 3-6: A sample dendrogram representing hierarchical clusters for point estimates shown in Figure 3-5.

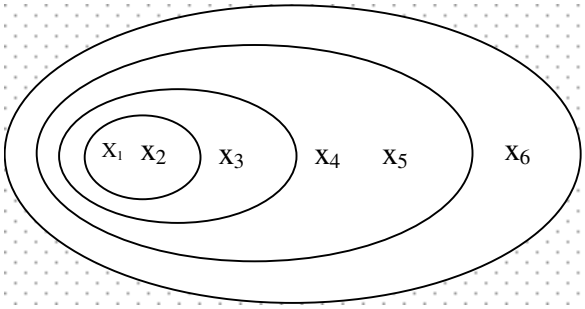


Figure 3-7: Nesting of the most optimum intervals as determined from the distance metric and cost function.

Using Definition 3-10, weights for each of the new intervals can now be determined from the original weights as:

$$m(C_i) = \sum_{x \in C_i} w(x) - \sum_{x \in C_{i+1}} w(x) \qquad \textbf{(3-32)}$$

Equation 3-32 can be interpreted as the weight that belongs only to that interval and not to any of its subsets. The process of combining measurements using the cost function yields a total of Q_H consonant intervals. The possibilistic weights for each of the nested intervals is then calculated from the derived weights as:

$$\pi(x) = \sum_{x \in C_j} m(C_j) \qquad \textbf{(3-33)}$$

Once nested intervals and the corresponding weights are derived, a possibility distribution is simply a trace over the intervals. Nesting of intervals as such is applicable when it has been determined that the data obtained is assumed to represent a single unknown quantity and the data converges to a true value. Even in cases where data is truly variable (as in variation in population characteristics) clustering techniques along

with possibility theory allow grouping of data to find possible values of the unknown quantities and thus offering significant benefits in data analysis at the explorative as well as advanced stages.

3.2.3.1.1 Outline of the Procedure

The general procedure for deriving a possibility distribution from point estimates thus includes:

1. Order the data and determine a measure of central tendency measure such as a geometric or an arithmetic mean or median. Since possibility distributions are inherently unimodal this measure determines the most possible interval.
2. Determine the weights of the original estimates, $w(x_i)$. The weights can be determined from the empirical data using conventional frequency analysis or any other appropriate method. These weights reflect the amount of evidence that is focused on singletons.
3. Calculate the distances using Equation 3-30 and organize data such that the data is sorted in the increasing order of the distance.
4. Determine the consonant intervals by aggregating observations that are least expensive (smallest difference in costs).

3.2.3.1.2 Example

Let the sample space consist of eight unique observations, $X = \{4, 5, 3, 2, 4.1, 5.2, 5.1, 1\}$. From the analysis of the data it can be seen that the anchor determined as the mean of the

data is, x_c=3.68. Let the weight assigned to each observation be determined using Equation 3-26. Using the mean value as the anchor and applying the Euclidean distance metric the distances calculated are shown in Table 3-6. From the results of Table 3-6, a graph as determined using the cost function is constructed as shown in Figure 3-8. The nodes are connected to the other appropriate nodes such that points with the least cost function are combined together.

Table 3-6: Sample singleton measurements and corresponding weights and the distance

Observation, x	Weights	δ [1]
4	1/8	0.32
4.1	1/8	0.42
3	1/8	0.68
5	1/8	1.32
5.1	1/8	1.42
5.2	1/8	1.52
2	1/8	1.68
1	1/8	2.68

[1]Distance as determined from Equation 3-30

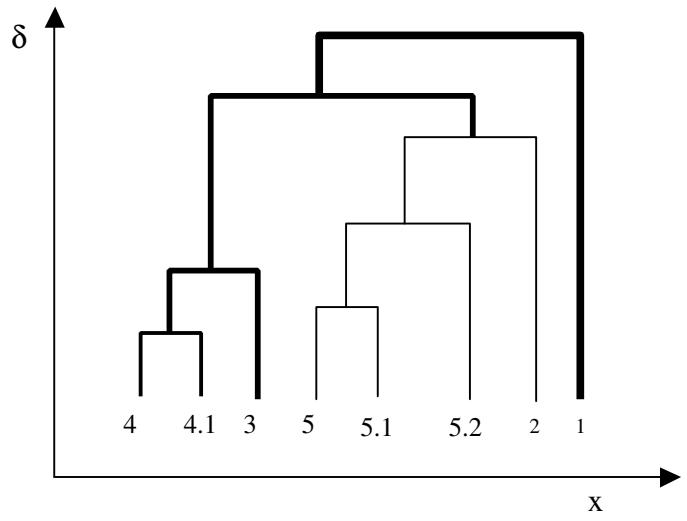


Figure 3-8: Dendrogram indicating the similarities between the measurements from Table 3-6.

Now, based on the distances in Table 3-6 and using Equation 3-30 and Equation 3-31, clusters are identified as:

$$C_1 = [4,4.1],\ C_2 = [3,4.1],\ C_3 = [2,5.2],\ C_4 = [1,5.2].$$

Finally, referring to Equation 3-32, the vector of weights for the nested intervals is:

$$w(C) = \{2/8, 1/8, 1/2, 1/8\}$$

Using these weights a possibility distribution is derived using Equation 3-33 and is shown in Figure 3-9.

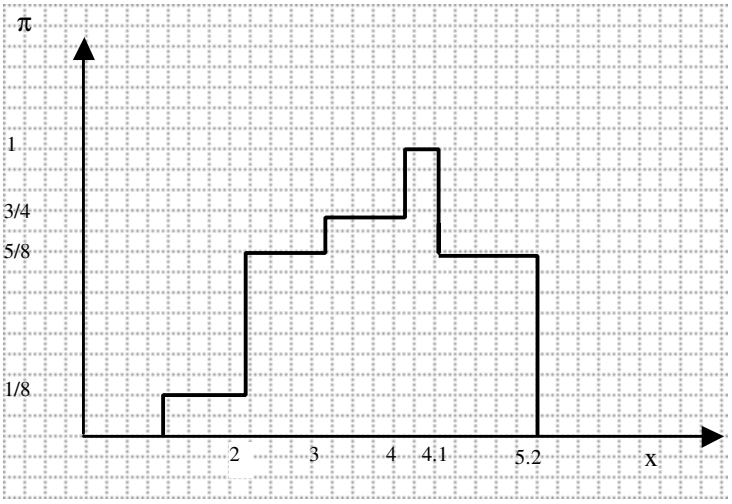


Figure 3-9: Possibility distribution depicting the possibilistic weights assigned to nested intervals.

3.2.3.2 Developing Possibility Distributions from Disjoint Measurements

When measurements are completely disjoint intervals as shown in Figure 3-10, a possibility distribution can be determined using principles similar to point estimates. Disjoint intervals pose the same challenge as for point estimates where each measurement is disjoint and no common interval can be determined. As explained in Section 3.2.3.1, clustering techniques offer mechanisms for deriving possibility distributions from such forms of measurements. Unlike point estimates however, intervals are considered to be two-dimensional in that two attributes are required to define a single interval. Hence when disjoint intervals are clustered, the distance and cost functions are based on a two-dimensional analysis.

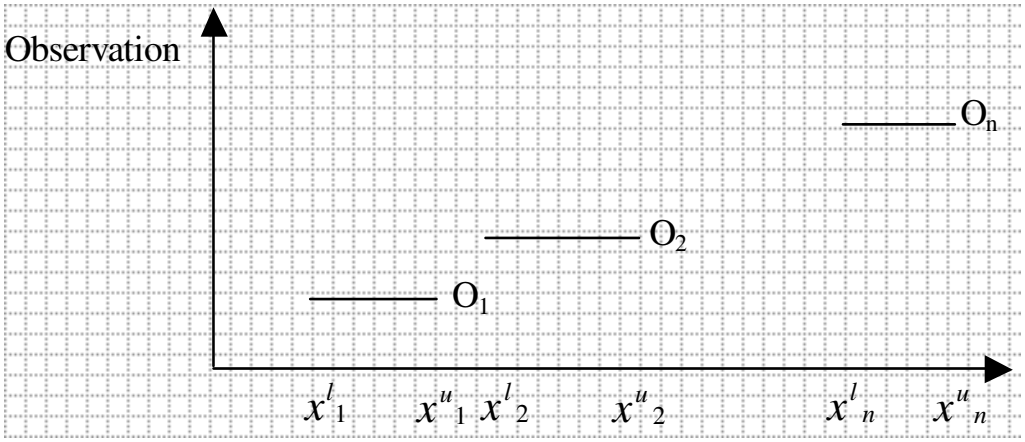


Figure 3-10: A graphical depiction of disjoint measurements.

Given *M* observations that are disjoint intervals,

$O_1 = [x^l_1, x^u_1]$, $O_2 = [x^l_2, x^u_2]$, ……., $O_M = [x^l_M, x^u_M]$,

such that, $\forall O_i, O_j, O_i \cap O_j = \varnothing$, weights assigned to each interval can be derived from conventional counting techniques as:

$$w(O_i) = \frac{n(O_i)}{M} \qquad \textbf{(3-34)}$$

where $n(O_i)$ is the number of times the measurement O_i has occurred.

The two bounds formed by the min on all the lower bounds of the intervals and the max of all the upper bounds include the entire set of measurements. Therefore, the support for the distribution can be determined as, $S = \left[\min_{x \in F}(x^l), \max_{x \in F}(x^u)\right]$, where x^l and x^u are the lower and upper bounds of the measurements respectively. An anchor (see Section 3.2.3.1 for a general description of an anchor) can be calculated from the collection of

measurements as a measure of central tendency:

$$X^a = \left[\frac{[x^l{}_1 + x^l{}_2 + ... + x^l{}_n)}{M}, \frac{[x^u{}_1 + x^u{}_2 + ... + x^u{}_n)}{M} \right] \quad \textbf{(3-35)}$$

The anchor determined from Equation 3-35, similar to that derived for point estimates, differs from the central tendency measures used in probabilistic analysis in that this measure is merely an approximate interval in which the most possible value would lie and not a parameter of a distribution. The distance metric that defines the distance between each measurement is then calculated from the anchor using the Euclidean norm and is given as:

$$\delta(O, X^a)_2 = \left(\sum_{d=1}^{2} \left(x_d - x^a{}_d \right)^2 \right)^{1/2} \quad \textbf{(3-36)}$$

The delta in the above function indicates that the distance is a sum of the distances between each attribute *d*, of the measurement O, and the anchor X^a. Once the distance metric is computed, the measurements can then be arranged in a sequence of increasing distances such that the measurements with the least distances are adjacent to each other. Eventually this measurement ordering is used to determine the cost function between adjacent intervals as:

$$\Delta = \min\left(\delta(O_i, X^a)_2, \delta(O_j, X^a)_2 \right) \quad \textbf{(3-37)}$$

The cost function given in Equation 3-37 determines the aggregation of the intervals such that the intervals with the least cost are aggregated together and the resulting new consonant intervals are determined as,

$$C_i = \left[\min\left(x^l \middle| x^l \in O_{New}\right), \max\left(x^u \middle| x^u \in O_{New}\right)\right], \quad \textbf{(3-38)}$$

where, O_{New} is the interval determined as the union of all the intervals that are aggregated. Once all the Q_H consonant intervals are determined as above, the weights and possibilities assigned to each interval are calculated similar to Equation 3-32 as:

$$m(C_i) = \sum_{O_i \in C_i} w(O_i) - \sum_{O_i \in C_{i+1}} w(O_i) \quad \textbf{(3-39)}$$

And finally the possibilistic weights are as defined in Equation 3-33, and are given as,

$$\pi(x) = \sum_{x \in C_J} m(C_j)$$

As can be seen from the above set of equations, disjoint intervals are a generalization of the point estimates.

3.2.3.2.1 Outline of the Procedure

1. Calculate the initial weights of each measurement according to Equation 3-34.
2. Determine a suitable anchor to approximately determine the most possible value. A simple anchor based on the central tendency measure as shown in Equation 3-35 can be used in most cases.
3. Order the measurements according to the distances as calculated using Equation 3-36 such that the interval that holds the highest possible value is on the top. This ordering will help determine the proximity of measurements to the most possible value and aid in the aggregation of the measurements.

4. Calculate the cost functions between each measurement and aggregate the measurements with the least cost function and determine the interval endpoints as given in Equation 3-38.

5. Compute the final weights of each interval and the possibilistic weights according to Equation 3-39 and Equation 3-33.

3.2.3.2.2 Example

Let the observations from a measuring device be given as, $O_1 = [1,2]$, $O_2 = [7,10]$, $O_3 = [3,5]$, $O_4 = [11,14]$, $O_5 = [6,7]$ (Figure 3-11). The weights assigned to each interval and the distance metrics as calculated from the anchor are shown in Table 3-7, where the anchor as calculated from Equation 3-35 is, $X^a = [5.6,7.6]$.

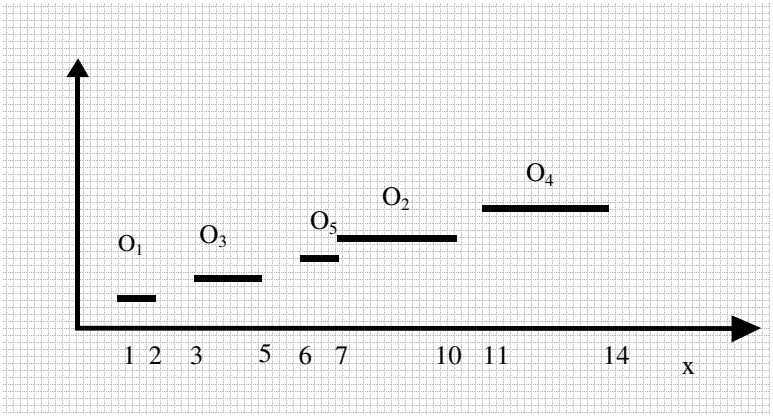


Figure 3-11: Graphical depiction of disjoint intervals for Example 3.2.3.2.2.

Table 3-7: Weights and distance metrics for Example 3.2.3.2.2

Measurement	Value	Weight	Distance metric δ
O_1	[1,2]	0.20	7.25
O_2	[7,10]	0.20	2.78
O_3	[3,5]	0.20	3.68
O_4	[11,14]	0.20	8.37
O_5	[6,7]	0.20	0.72

Arranging the measurements in the increasing order of the distance measure yields the following order: $O_5 > O_2 > O_3 > O_1 > O_4$ and Figure 3-12 shows the aggregation of the measurements resulting from the application of the cost function.

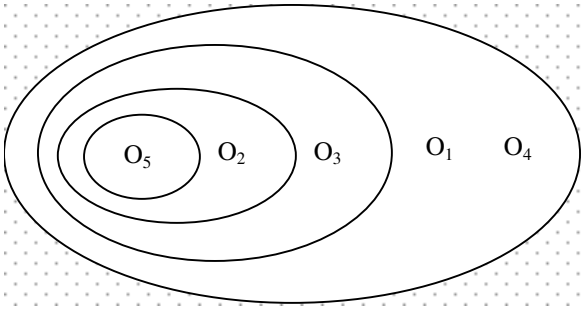


Figure 3-12: A graphical depiction of the nested intervals for Example 3.2.3.2.2.

The nested intervals are calculated from Equation 3-38 as:

$C_1 = [6,7]$, $C_2 = [6,10]$, $C_3 = [3,10]$ and, $C_4 = [1,14]$. And the corresponding weights of the nested intervals as determined from Equation 3-39 are, $w_1 = 0.2$, $w_2 = 0.2$, $w_3 = 0.2$

and, $w_4 = 0.4$. Resulting possibilistic weights are simply a summation of these weights and are shown in Figure 3-13.

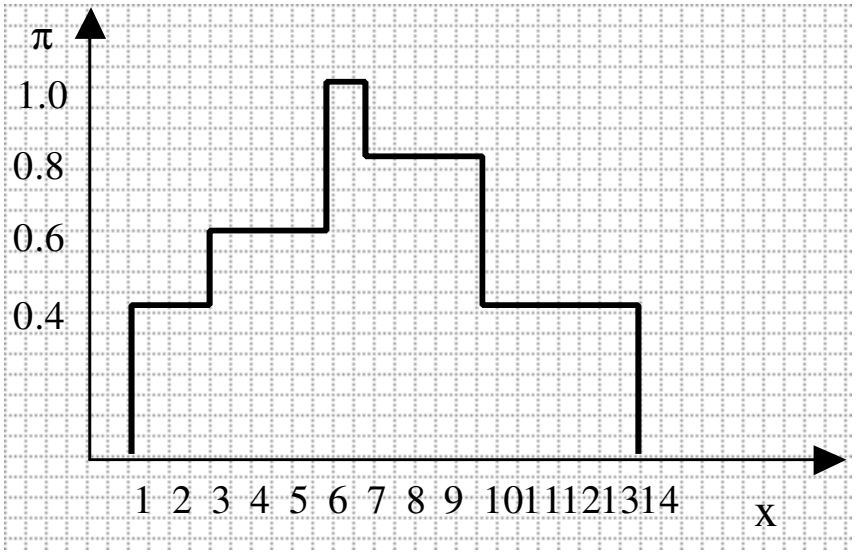


Figure 3-13: Possibility distribution function for the final nested intervals extracted from disjoint measurements.

3.3 Propagation of Possibility distributions

Though the development of possibility distributions as described in the previous section is crucial, the exercise is futile without the sound framework of using them. The fundamental concepts of possibility theory mixed with fuzzy logic, fuzzy arithmetic and interval analysis provide a rich and flexible framework for using possibility distributions in a number of unique situations where data is severely lacking or is ambiguous [Dubois and Prade, 1988]. Much the same way as the use of a probability distributions differs according to the context in which they are used, possibility distributions are to be used according to the uncertainty they are to represent. Possibility distributions are used to

represent uncertainty in parameters that are part of a model or as stand-alone distributions from which decisions are made. For example, a possibility distribution that represents the possible values of contaminant concentration at a specific location L can be used to determine the risks from exposure to the contaminant by including the distribution of the contaminant in a risk assessment model or it can be used to determine the possible values of the contaminant concentration to aid in decision-making models that determine if remediation is necessary. Given a possibility distribution, an analyst might be interested in answering the following questions:

- What are the possible concentrations of agent A at location L?
- What are the possible risks from exposure to agent A?
- If the concentration is beyond the regulatory limit what are the possible costs of implementing various remedial alternatives?
- What is the possibility of component C failing?
- What is the possibility of system S failing?

Literature in the field of fuzzy logic is sprinkled with answers to the above questions where the distributions are considered to be fuzzy membership functions. The underlying similarity of axioms of fuzzy logic to those of possibility theory makes the use of possibility distributions in similar situations a natural extension. Answers to the first two questions can be classified as falling under the domain of human health risk assessments, the answer to the third question is usually addressed by risk managers while the answers to the last two questions are addressed in the field of reliability of systems.

In reliability analysis, where fault trees and event trees determine the failure

sequence of events, the component failure rates represented as possibility distributions are combined at various levels using T-norms and co-norms [Dubois and Prade, Gupta and Qi, 1991]. See also Lucero [1999] for more on use of T-norms and co-norms in fault trees. These are logical combinations that are used to calculate the union, intersection and complement of two or more possibility distributions. The most common norm used is Zadeh's min and max norms, where min determines the intersection of two or more fuzzy membership functions and max determines the union. These are analogous to the product and sum norms used to determine the intersection and union of probability measures. As opposed to the logical combination, functional propagation of possibility distribution involves the use of fuzzy arithmetic and interval analysis. Methods to propagate possibility distributions in either setting are well developed [Ross 1995] and some have been summarized in Chapter 2 of this dissertation. Since possibility distributions are functions defined on intervals, interval analysis is considered be a preferred approach for functionally propagating the distributions.

Possibility distributions in the form developed here can also be used in modeling knowledge-based systems where the outputs are related to the inputs by an unknown function and are effectively represented as fuzzy "If-Then" rules. For example, when the relation between the magnitude and the effect of exposure to a chemical X is only known partially, "If-Then" rules can be formulated to form a knowledge-base that includes all the measured concentrations as premises and the observed outcomes of effects as the consequences. Once such a knowledge-based system has been established specific rules can be fired according to the value of a premise. An actual application of a rule-based

system for fuzzy membership functions has been applied in soil infiltration models [Bardossy and Disse, 1993].

Chapter 4 From Theory to Practice

The purpose of this Chapter is to illustrate the application of methods developed in Chapter 3 to data extracted from the risk analysis literature.

Risk assessments are made up of parameters that are seldom known precisely and the evidence that is available to support specific values for these parameters is usually incomplete. There is extensive data available in specific areas of risk assessment (such as dose-response assessment and structural information for specific chemicals); however owing to their highly multi-disciplinary nature, risk assessments are prone to including variables that are incompletely known. Even in cases where there is ample information for deriving uncertainty parameters, the final outcome can be ambiguous as the experiment itself is sometimes interposed with noise or is accompanied by human judgments. The level of noise and amount of subjective human inputs into the models usually varies according to the complexity and availability of information and its effect is seldom known with precision. For example, when determining the effects of chemicals on human health, values of exposure concentration derived from fate and transport models are only approximate. Moreover, the fate and transport models are combined

with data from dose-response models, and both of these types of models are only as thorough as the knowledge about system they represent.

In such cases it is more judicious to combine what is known from tests (precise values determined from testing a few samples that are representative of the system) with what is not known about the heterogeneity of the system and the model (inherent uncertainty in the system and model). As introduced in Chapter 2, previous uncertainty methods used to model uncertainty in risk assessments are limited by their scope and do not offer a scheme that allows for a natural representation of all forms of uncertainty. An efficient solution to modeling uncertainty might be to blend probability theory with other theories so risk analysts have all the tools at their disposal when conducting risk assessments. Such an approach is proposed here in this dissertation.

4.1 Case Study: Risks from Exposure to Residential Radon

Radon is an odorless, tasteless, invisible radioactive gas that is prevalent in air, soils, and many residential buildings. Radon (chemical symbol Rn) is a naturally occurring radioactive gas found in soils, rock, and water throughout the U.S. It has numerous different isotopes, but radon-220, and -222 are the most common. Radon-222 is the decay product of radium-226. Radon-222 and its parent, radium-226, are part of the long decay chain for uranium-238. Since uranium is essentially ubiquitous in the earth's crust, radium-226 and radon-222 are present in almost all rock, soil, and water. Radon causes lung cancer, and is a threat to health because it tends to collect in homes, sometimes to very high concentrations. As a result, radon is the largest source of

exposure to naturally occurring radiation. It has been determined that exposures to sizeable concentrations of radon can result in a significant increase in lung cancers in humans [EPA, 2002].

Indoor radon gas exposure is one of the leading causes of lung cancer in the United States. In some areas of the country, as many as one out of two homes has high levels of radon. Radon levels can soar during the colder months when residents keep windows and doors closed and spend more time indoors. Levels of radon gas in the air that are considered by the EPA to be a cause for concern are 4 pico Curies per liter (pCi/L) or higher. Hence, the study of radon gas in human health risk assessment is an *important problem*.

The incidence of higher rates of lung cancers in uranium miners and the presence of high concentrations of radon gas in mines has resulted in numerous epidemiological studies to determine the effects of radon exposure to human health. These miner studies demonstrated a significant relationship between radon and lung cancers [Lubin *et al*., 1994]. It is now estimated that indoor radon can cause anywhere between 15,000 to 22,000 lung cancer deaths annually in the US [EPA, 2002]. Extrapolation of results from miner studies to those that are commonly prevalent in residential settings resulted in EPA regulating radon at concentrations of 4 pCi/L (1 curie being equivalent to the quantity of radioactive material yielding 3.7 x 10^{10} disintegrations per second). Though it is established that increased exposures to radon causes cancer, the direct extrapolation of data from miners to residential population has been questionable due to the differences in the characteristics between the miners and the general population and differences in the environmental conditions in the mines and in the residential buildings. These differences

prompted further epidemiological studies that studied the effects of radon exposures on human populations in residential settings. Over the past decade many studies were conducted in various parts of the world [Field, 2001]. Each of these studies has determined a link between radon exposure and lung cancer, however, the intensity of the effect has varied from study to study [Lubin and Boice, 1997; Field, 2001].

In the following case studies we explore the application of possibility distributions to model uncertainties in individual epidemiological studies and we also explore the use of possibility theory in a meta-analysis study as conducted by Lubin and Boice [1997], and in a risk study conducted by Field *et al*. [2002].

4.1.1 Deriving Possibility Distributions from Meta-Analysis of Pooled Studies

When faced with a limited amount of data, and the constraints of time to collect further data, one of the effective methods of data analysis is to identify data from multiple studies and pool them to extract more informative patterns. Such analysis of data from multiple studies is sometimes referred to as meta-analysis and is especially useful in areas such as epidemiological and environmental studies wherein data are hard to acquire and a single study scarcely presents any conclusive evidence. Meta-analysis has been defined as the "statistical analysis of a large collection of results from individual literature for the purpose of integrating the findings" [Glass, 1976]. There are a number of methods available to conduct meta-analysis and more details are presented elsewhere [Thacker, 1990]. Meta-analysis of pooled data is conventionally accomplished using a combination of statistical analysis, regression methods, and likelihood models. This form of statistical

pooling works well when the uncertainties underlying each study have been confidently tackled and each study presents a precise value to the pooling model.

However, due to the inherent nature of epidemiological studies such precise estimates are seldom identified in each study. The nature of the results depends on the uncertainty analysis model used within each study, and can be ambiguous, vague and/or non-specific. Another challenge of conducting meta-analysis on studies extracted from literature is that there is usually inter-study and intra-study variability due to factors that are usually not identified by the studies and therefore are not reflected in the final estimates published. Finally, as meta-analysis is performed on controlled studies the nature of variation in the estimates between studies is non-random. Due to these reasons, meta-analysis studies could benefit from uncertainty models that are more amenable to summarization of the data in *weaker terms* (such that measures do not necessarily add to 1.0) and more natural (consistent with human thought process) in the data representation. This case study attempts to exemplify the usefulness of possibility theory to meta-analysis. Generation of possibility distributions as discussed in the earlier chapters allows one to systematically uncover relevant empirical information and to naturally represent the uncertainty in the entire range of data as derived in the original studies.

To illustrate the possibilistic model's application, a meta-analysis study conducted by Lubin and Boice [1997] was used. In their study, Lubin and Boice pool lung cancer data from eight residential radon gas exposure studies. The goal of the study was to estimate the relative risk values and their corresponding 95% confidence intervals (CI) from the pooled studies. Relative risk estimates were determined by fitting a log-linear

model to each of the original 8 studies (See Appendix A for the fitted curves):

$$\log[RR(x, x_o)] = \beta(x - x_o) \qquad \textbf{(4-1)}$$

In Equation 4-1, x is the exposure dose, x_o is the referent category and β is the slope of the exposure-response curve. The choice of the model and its derivation are explained elsewhere [Lubin and Boice, 1997] and it is sufficient for this case study to understand the basic definitions of relative risk (RR) and slope factor (β). Relative risk is defined as the ratio of the risk rates in those exposed to the stressor to those unexposed and determines the excess risks from exposure to the hazard. The factor β is used to relate the relative risks to exposures and determines the rate of increase in relative risk for a unit change in the exposure concentrations. In a typical radon epidemiological study, relative risks or odds ratio are calculated from very few categories of exposure concentrations, with the first category usually referred to as the "referent category". In most cases such few sampling categories is not sufficient to determine the slope factor accurately, and this is especially true for radon studies where there is weak association between the dependent (response) and the independent (exposure) variable. The utility of a meta-analysis study is to determine the exposure-response factor (β) from a collection of studies so that the pooled studies can extract stronger statistics.

In their meta-analysis study Lubin and Boice determine β by first calculating estimates β_i, $i = 1,....,8$ for each study by fitting Equation 4-1 to the estimates in each individual study and then obtain the overall β as the mean value of the probability distribution formed by β_i. The underlying probability distribution is assumed to be a

normal distribution. In our case study, we do not need to assume any distribution on β_i s, nor to derive a possibility distribution of the relative risk estimates calculated using individual β_i. Rather, the concepts of possibility theory allow for the interpretation of the state of uncertainty related to the variable under study. The variable considered in this case study is the relative risk and a possibility distribution is derived for the 95% confidence intervals of the relative risks determined using the β_i s from each study (these are shown in Table 4-1). The 95% confidence interval is assumed to reflect the variability of the data within a study, and the range of the interval is considered to represent the ambiguity of the expert in picking any one value for the relative risk.

Since the possibilistic meta-analysis does not necessarily assume inter-study randomness, the randomness that is inherent in the 95% confidence interval estimates of each study is transferred into a possibility distribution. Therefore, in addition to randomness, ambiguity is also represented. The resulting distribution can be assumed to represent second-order uncertainties, where, now a possibility distribution represents the ambiguity in the 95% confidence interval values of relative risks. We also illustrate how possibility distributions can be generated to represent the ambiguity in the relative risk values arising from point estimates where the ambiguity is not presented in the measurements themselves but is, that which occurs due to lack of data to accurately portray uncertainty as that which is only due to randomness.

4.1.1.1 Deriving Possibility Distribution from the 95% Confidence Intervals

Due to a lack of complete data from the original studies used by Lubin and Boice, data from their tables are used without any modifications. Table 4-1 shows these

values.

Table 4-1: Estimates of Relative Risk at 150 Bq/m3 and 95% confidence intervals for each study and for all the studies combined (Lubin and Boice, 1994)

Study	Relative Risk	95% CI
Finland – I (F-I)	1.3	1.09-1.55
Finland-II (F-II)	1.01	0.94-1.09
New Jersey (NJ)	1.83	1.15-2.90
Shenyang (Sh)	0.84	0.78-0.91
Winnipeg (Wpg)	0.96	0.86-1.08
Stockholm (Shlm)	1.83	1.34-2.50
Sweden (Swe)	1.20	1.13-1.27
Missouri (MO)	1.12	0.92-1.36
Combined	1.14	1.01-1.30

Assigning weights to each study

Each of the studies is assumed to be equally accurate and hence equal weights are assigned to each one. These weights represent the evidence available to support the statement that the relative risk values are between the ranges indicated by the interval. In reality, some studies can be more extensive than others or the characteristics of the study can resemble the intent of the meta-analysis more closely and in such cases the weights can be adjusted to reflect the confidence of the analyst in the study. Lubin and Boice assign weights to each study by using a parametric approach prescribed in the literature [Whitehead and Whitehead, 1991] where the asymptotic variance in the study defines the value of the weight.

Derivation of most possible interval

Determining the consonant intervals by taking the intersections of the original set-valued observations is accomplished according to the procedure outlined in Chapter 3. The support set that includes all the possible values from for all the studies is given as, S

= [0.78, 2.9] and the average of the newly formed intervals is [1.08, 1.44]. Figure 4-2 shows all the intersections along with the support set and the average interval. Intersecting intervals derived from all the intervals as shown in Figure 4-1 are tabulated in Table 4-3 along with the normalized weights. The columns labeled "lower" and "upper" represent the lower and upper bounds of the new interval, respectively. The Min-Max row represents the support of the possibility distribution and represents the set derived from the union of all the focal elements. The weight assigned to this interval is optimistically taken as the minimum of all the weights by assuming that at least one of the values in the support has to be least possible when compared with the most possible value (most possible value being the value with complete possibility).

Table 4-2: Weights assigned to each study's interval based on the assumption that the studies are equally reliable

Study ID	Interval*	Weights
F-I	[1.0900, 1.5500]	0.25
F-II	[0.9400, 1.0900]	0.25
NJ	[1.1500, 2.9000]	0.25
Sh	[0.7800, 0.9100]	0.25
Wpg	[0.8600, 1.0800]	0.25
Shlm	[1.3400, 2.5000]	0.25
Swe	[1.1300, 1.2700]	0.25
MO	[0.9200, 1.3600]	0.25

*Intervals based on 95% CI estimates, however subjective and objective interval derived from other criteria can be readily used

The resulting intervals (Figure 4-2) are then analyzed for consonance by choosing the interval set that could contain the most possible values. Other methods such as entropy measures and plausibility traces [Chavez, 2002] are appropriate when the number of intervals is very limited, however, the mean value is considered to be intuitive and effective in determining consonance in most cases. In addition, an expert's knowledge

can also be used to determine the initial anchor. The intent of this dissertation is, however, to develop purely empirical distributions and hence strictly quantitative measures such as the mean value were employed.

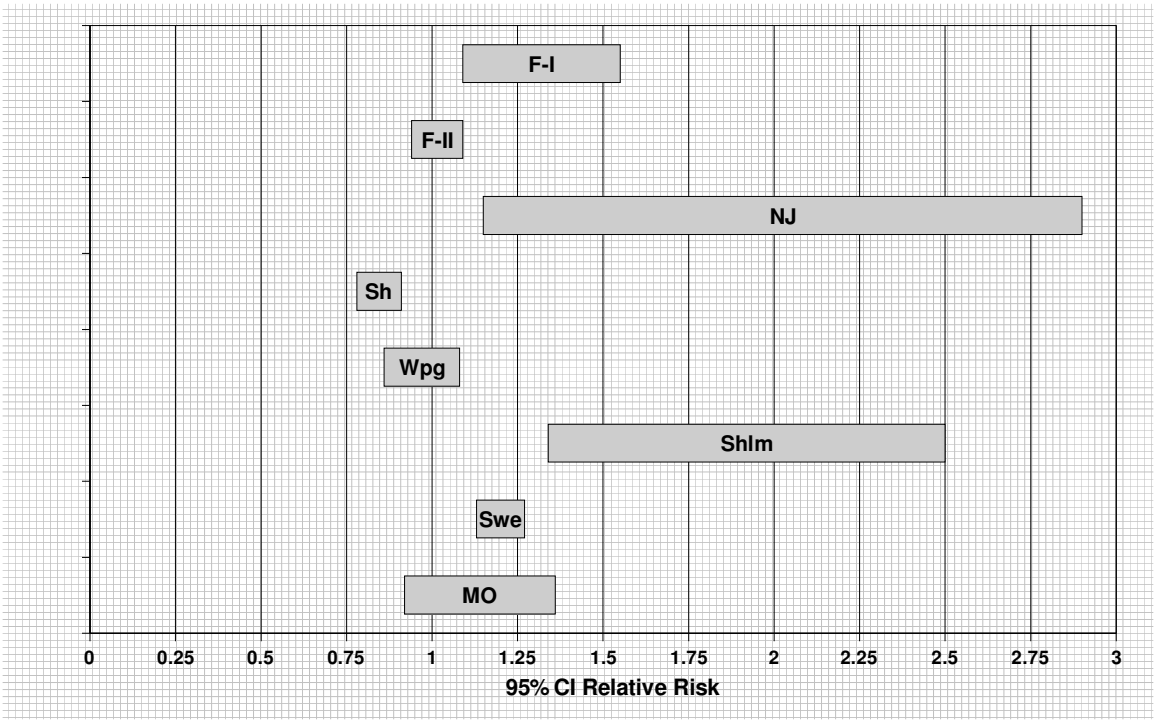


Figure 4-1: 95% confidence intervals on relative risks at exposure levels of 150 Bq/m^3 (1 pCi/L = 37 Bq/m^3).

Conceptually, the meaning of the anchor that determines the most possible interval differs from the meaning of the mean values that define a probability distribution, in that the latter is based on the assumption of randomness and the stricter axioms of probability theory while the former does not assume any randomness and supports a weaker axiom of possibility theory that the possible value lies within some range of values. Hence, the interval of [1.08, 1.44] is simply asserting that the relative risk value can lie somewhere between the values 1.08 and 1.44. This is a weaker claim than indicating that the values between 1.08 and 1.44 are known to occur in 50% of the cases,

i.e., $P(1.08 \leq X \leq 1.44) = 0.5$ and that $P(\neg X) = 1 - p = 0.5$.

Table 4-3: Intervals obtained by intersection of the original set of observations

	Lower	Upper	Wt[1]	Normalized Wt
1	1.34	2.50	0.13	0.05
2	1.34	1.55	0.13	0.05
3	1.34	1.36	0.13	0.05
4	1.15	2.90	0.13	0.05
5	1.15	1.55	0.13	0.05
6	1.15	1.36	0.13	0.05
7	1.15	1.27	0.13	0.05
8	1.13	1.27	0.13	0.05
9	1.09	1.55	0.13	0.05
10	1.09	1.36	0.13	0.05
11	1.01	1.30	0.13	0.05
12	0.94	1.09	0.13	0.05
13	0.94	1.08	0.13	0.05
14	0.92	1.36	0.13	0.05
15	0.92	1.08	0.13	0.05
16	0.86	1.08	0.13	0.05
17	0.86	0.91	0.13	0.05
18	0.78	0.91	0.13	0.05
Min-Max	0.78	2.90	0.13	0.05
Average	1.08	1.44		
		Sum	2.38	1.00

[1]Weights for each intersecting interval are determined by applying the min norm across the weights of the original intervals.

Rather than asserting such precise statements defined by crisp boundaries, a risk analyst might prefer to assert their confidence in varying ranges of values that converge to the most possible value. Therefore, the mean value in the possibilistic model is an assertion that the most possible value lies in an interval that is closest to the mean and no other precision is assumed. Lack of precise statements in this case is not a reflection of inaccuracy but merely a reflection of the realities of the data.

In the case study, using this mean-value technique resulted in consonant intervals shown in Table 4-4. An inspection of the consonant intervals reveals that the weights assigned to each interval reflects the number of observed values that are known to occur within each interval. Hence, though the most possible relative risk values are between 1.15 and 1.27, the weight of 0.13 assigned to this interval reflects the actual weight an analyst is comfortable in assigning only to this interval and to no other interval, and similarly the weight assigned to interval [0.78, 2.9] is 0.45 indicating that the evidence strongly supports just this interval and no other interval. As seen from Figure 4-2, the higher the number of original intervals included within a derived consonant interval the higher is the weight assigned to it.

Discussion

A comparison of the possibility distribution and the 95% confidence interval obtained by Lubin and Boice is shown in Figure 4-3. Lubin and Boice assume that the exposure-response parameter, β, varies randomly between the studies and the contribution of each β to the overall estimate is weighted by a weight equal to the inverse of the variance within each study.

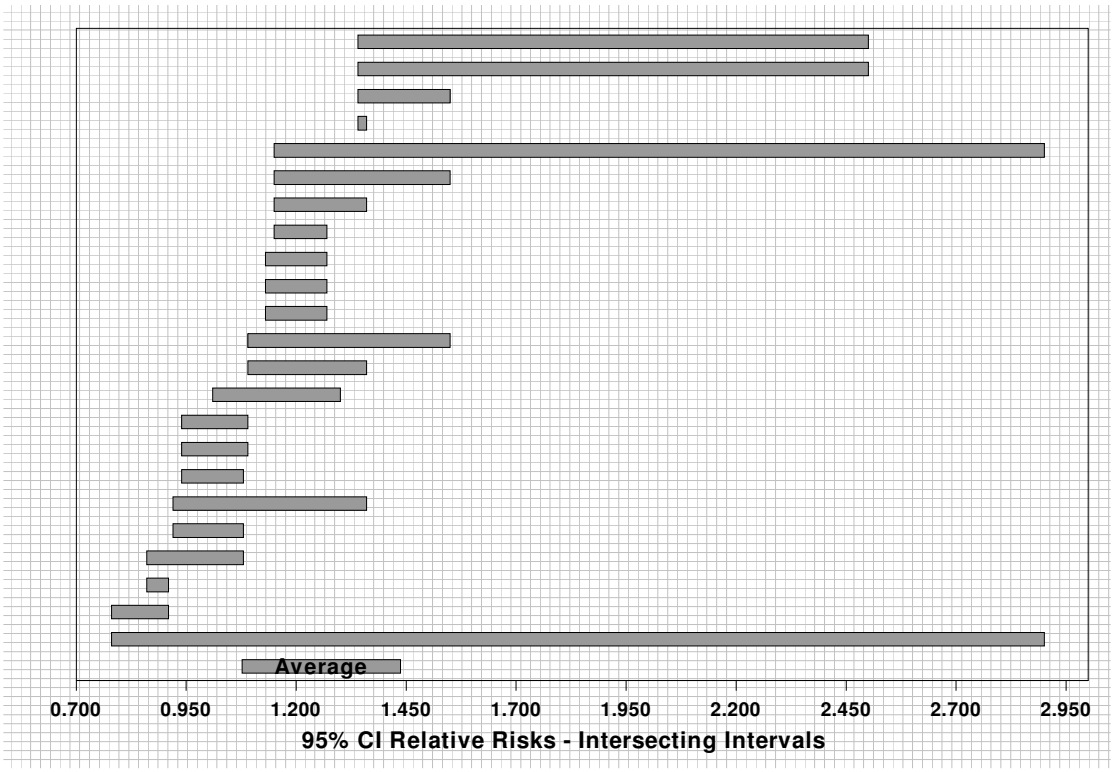


Figure 4-2: Intersecting intervals from the original set of intervals. A set that is the union of all the intervals A=[0.78,2.9] forms the support of the distribution and the average of all intervals (Average = [1.08, 1.44]) is used as the anchor for determining the most possible interval.

Table 4-4: Redistribution of weights from the non-consonant intervals to the consonant intervals

	Consonant Intervals	Wt[1]	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Final Wts.
		Redistribution weights															
1	[1.15, 1.27]	0.05	0.00	0.00	0.23	0.27	0.31	0.26	0.00	0.00	0.34	0.00	0.00	0.00	0.00	0.00	0.13
2	[1.13, 1.27]	0.05	0.00	0.00	0.20	0.23	0.27	0.26	0.00	0.00	0.32	0.00	0.00	0.00	0.00	0.00	0.12
3	[1.09, 1.36]	0.05	0.05	0.12	0.18	0.21	0.24	0.28	0.00	0.00	0.19	0.00	0.00	0.00	0.00	0.55	0.15
4	[1.09, 1.55]	0.05	0.15	0.72	0.20	0.24	0.14	0.16	0.00	0.00	0.11	0.00	0.00	0.00	0.00	0.14	0.15
5	[0.78, 2.90]	0.05	0.81	0.16	0.19	0.05	0.03	0.04	1.00	1.00	0.04	1.00	1.00	1.00	1.00	0.31	0.45

[1]Weights on the consonant intervals

In deriving the possibility distribution, however, no randomness or weighting is assumed (however this could be very easily included by varying the weighting factor) but rather, the 95% confidence intervals from each study are considered without any modifications. Hence the possibility distribution, in addition to extracting the most possible values, also represents the entire variation in the possible relative risk values. Lubin and Boice's estimates show that the resulting 95% confidence intervals for relative risks are [1.01, 1.30]. Due to the difference in the actual weights given to each study, the final estimates may vary numerically, however, the intent of the comparison is not to highlight the accuracy of the estimates, but to point out the difference in the meaning of the values derived from possibilistic and probabilistic analysis. By the definition of a 95% confidence interval based on the classical interpretation of probability, this suggests that the interval [1.01, 1.30] is just one of the intervals in a large number of repeatable studies that can contain the true value. Therefore, a 95% confidence interval in this case does not reveal much but to mathematically reduce the range of true values. Given the data in Table 4-1 and given that the variability within studies is large, how confidently can an analyst now answer the question, with what confidence can you advise that the true value of relative risk is within this smaller range of [1.01, 1.30]? The answer would probably be something other than 95%. An analyst in this case might be more comfortable in giving answers that are weaker or are more vague and ambiguous. The possibility distribution curve shown in Figure 4-3 offers a more realistic picture of the variability in the studies and offers an analyst a wider range of choices.

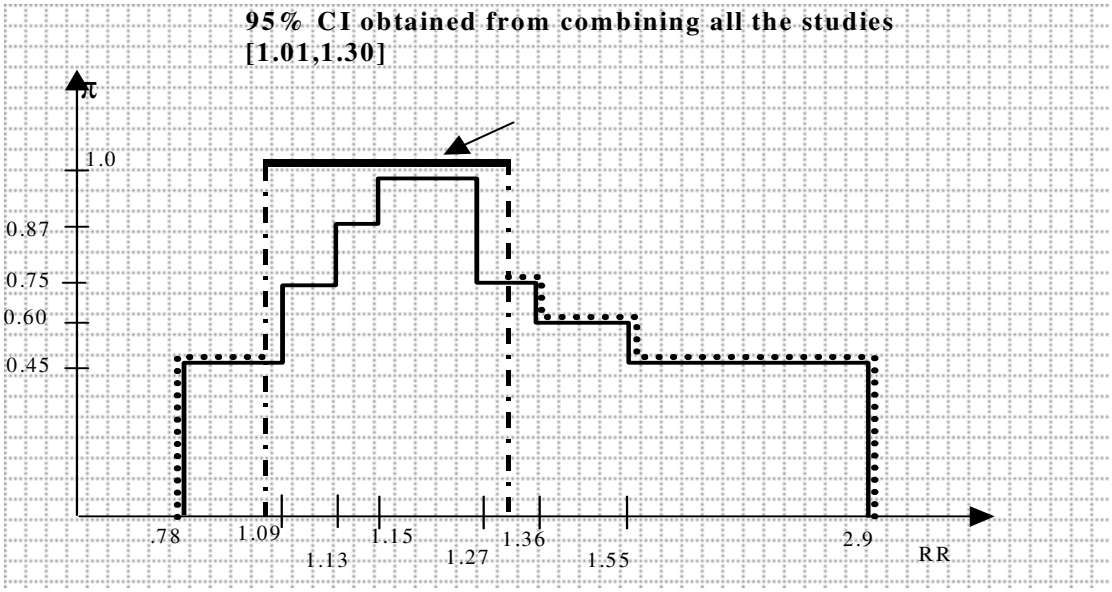


Figure 4-3:Possible relative risk values from the meta-analysis of eight case-control lung cancer residential radon exposure studies.

The same question posed to an analyst equipped with the possibility distribution can now be answered in a slightly different manner. If the question is, "with what confidence can you predict that the relative risk will lie between [1.01, 1.30]?", then the answer would be 25%. This response is a result of determining the certainty that the value will lie in the interval. The *certainty* value is determined through the concept of a necessity measure as defined in the previous chapter. Necessity is the dual of possibility and is defined by Equation 3-8 as $N(A) = \inf(1-\pi(x)|x \notin A)$, and it determines the degree to which one fails to doubt the occurrence of A = [1.01, 1.30]. It can alternatively be expressed as, $N(A) = 1-\Pi(\overline{A})$. Therefore, given this equation and referring to Figure 4.3, $N(A) = 1-0.75 = 0.25$. The dotted line in Figure 4.3 shows the region where events other than A are possible and hence the negation of this value indicates the evidence that completely supports A. Thus the *certainty* with which it is known that A will occur is

0.25 while the possibility it won't occur is 0.75 or, alternatively, it is completely possible that the event *can* occur.

It can also be seen from Figure 4.3, that an event is only certain to occur ($N(A) > 0$) when it is completely possible $\Pi(A) = 1$. This indicates that an analyst is only certain about the event occurring only if evidence completely confirms its occurrence. Therefore, a necessity value can also be considered the degree of confirmation when the value is completely possible. By providing these two dual measures, possibility theory enables an analyst to offer more realistic analysis of the uncertainty.

4.1.1.2 Deriving possibilistic curve from considering point estimates of the study

Possibility distribution for point estimates is derived using the second column of Table 4-1. Results of the analysis are shown in Table 4-5 and Figures 4-4 and 4-5. The values considered for point estimates are the relative risk values that were determined using the relative risk model given by Equation 4.1. Once again equal weights were assigned to each study by assuming that the studies were equally reliable. An anchor value of 1.26 was determined by taking the mean of the relative risk estimates from all the studies. The hierarchy of aggregation of the relative risk estimates as determined from the distance to this anchor is shown in Figure 4-4. The cost function as given by Equation 3-28 is used to select the most optimum points to aggregate. Bold lines in Figure 4-4 show the most appropriate order of aggregation. Table 4-6 shows the resulting nested intervals along with the interval weights and the possibilistic weights as determined by Equations 3-32 and 3-33. A possibilistic curve for the nested intervals

shows that the relative risk values with highest possibility lie between 1.2 and 1.3 indicating that it is very likely that radon is a carcinogen at an exposure concentration of 150 Bq/m^3 (since the relative risk value is greater than 1), with relative risk values of 1.83 also possible (Figure 4-5).

Table 4-5: Relative risk values and the distance between the point estimates determined from each radon lung cancer study [Lubin and Boice, 1994]

Study	Relative Risk	$m(x_i)$	δ
Finland – I (F-I)	1.3	1/8	0.04
Finland-II (F-II)	1.01	1/8	0.25
New Jersey (NJ)	1.83	1/8	0.57
Shenyang (Sh)	0.84	1/8	0.42
Winnipeg (Wpg)	0.96	1/8	0.30
Stockholm (Shlm)	1.83	1/8	0.57
Sweden (Swe)	1.20	1/8	0.06
Missouri (MO)	1.12	1/8	0.14

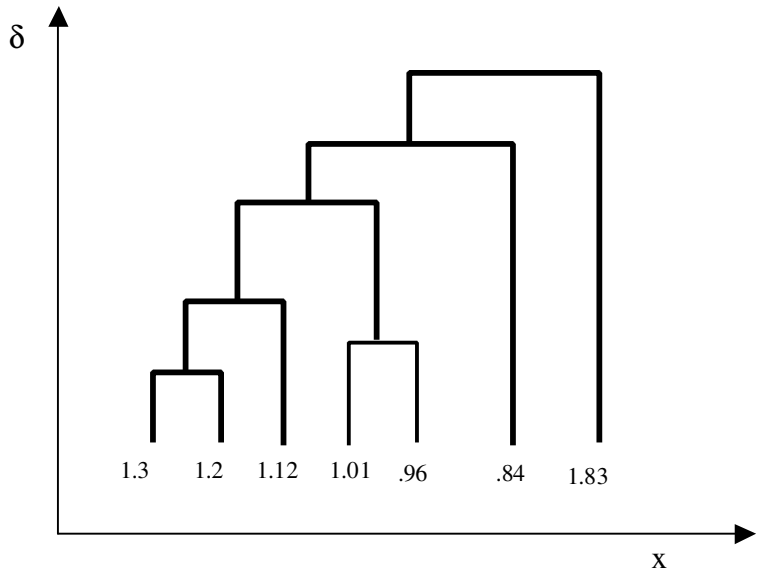


Figure 4-4: Nested interval structure of the relative risk point estimates.

Table 4-6: Final possibilistic weights determined from the nested intervals as shown in Figure 4-4

Interval (C_i)	Weight $w(C_i)$	Possibilistic weights $\pi(C_i)$
[1.2, 1.3]	0.25	1.0
[1.12, 1.3]	0.125	0.75
[0.96, 1.3]	0.25	0.625
[0.84, 1.3]	0.125	0.375
[0.84, 1.83]	0.25	0.25

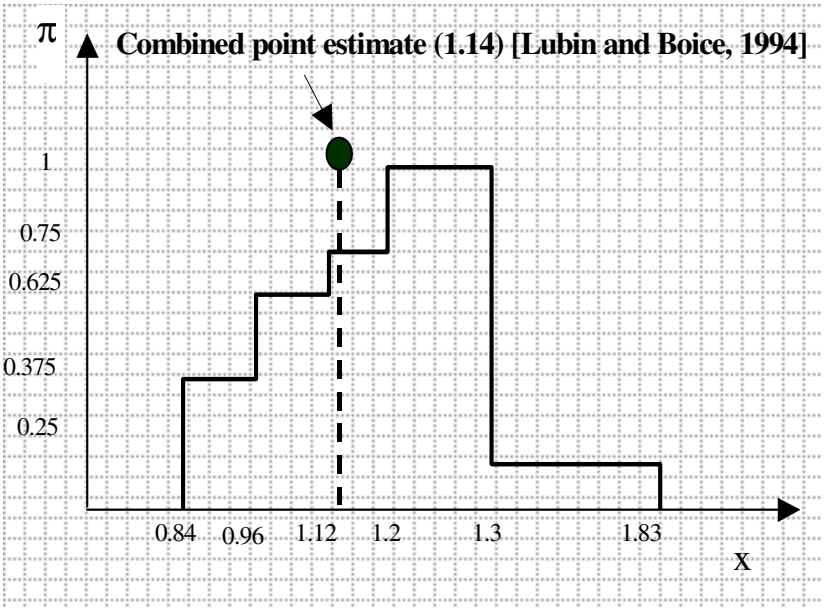


Figure 4-5: Possibility distribution derived from the relative risk point estimates.

In contrast, the point estimate value derived from Lubin and Boice [1994] shows that the excess risk is only 1.14. The expected difference is obvious since the derived possibility distribution is based on the inclusion of all the values, while that of Lubin and Boice are based on a weighted combination of the β from each study. Our main contribution is however that the possibility distribution developed, even though it is based purely on data and no assumptions, resulted in values that were consistent with those that resulted from the work of Lubin and Boice. The possibility distribution derived by considering the similarities of the values is more practical in a setting where the parameters of randomness are not known and the true variation within studies or the heterogeneity is suspect.

In the original meta-analysis study, the estimates generated were based on the assumption of a normal distribution wherein the underlying evidence is assumed to be precise in not only the calculation of the parameters of the distribution but also in the calculation of the variance of the slope factor β. As illustrated in Section 4.1.1.1, the possibility distribution constructed over the nested intervals also allows an analyst to make judgments at various levels of confidence without assuming any extra evidence. In comparison to the point estimates derived by Lubin and Boice where the estimate is 1.14, the most possible values derived from the use of possibilistic modeling range between 1.2 and 1.3, with a 25% possibility that the values could range between 0.84 and 1.83. From a cursory glance of the data in Table 4-1, it is easy to see that the range is 0.84 and 1.83, however what makes the possibility distribution interesting is that one can quantify the

possibility of various values of risks. For example, if one desires to know what the possibility of risks less than 1 is, the answer would be 62.5% (from Figure 4-5 where the interval along the abscissa is [0, 1.0]). However, unlike probabilistic analysis this does not constrain an analyst to estimate the possibility of values to be outside this range to be 37.5%. The possibility of the values being outside this range is in fact 100% (because the most possible value lies in this range).

Such calculations, in addition to allowing an analyst to judge the amount of ambiguity and vagueness in data, also helps compare risks for risk management decisions. For example, suppose the risks from two hazards are available and possibilistic analysis of the available evidence shows that the possibility of risk being greater than 1.0 in both cases is 100%. And, suppose that the possibility of risk being less than 1.0 is 60% in the first case and 65% in the second case, then it would be more prudent to work on reducing the risks from the first hazard. In a complex risk assessment and management case, the number of variables involved is usually significantly higher and quantification of these variables in such forms allows one to conduct analysis even when data is limited and only vague estimates are available.

In a more recent comprehensive radon study by the University of Iowa [Field, 2002], the relative risks from exposure to residential radon ranged from 1.24 to 1.83. This range is beyond the 95% confidence interval range of Lubin and Boice study but within the range determined by possibilistic analysis. Though it might have been coincidental that the range of actual values obtained from each original study of the meta-analysis study is between 0.84 and 1.83, reporting the entire range of possible values through possibility distributions, presents a more comprehensive summary of the results

for subsequent comparisons. This is especially more relevant when the data are very limited and there is no known scientific model that justifies precise relationships between dependent and independent variables. Though probabilistic and possibilistic methods represent different concepts, this comparison is shown to indicate the utility of the possibility distribution in deriving representative values from sparse data. While complicated probabilistic models exist, it seems more *natural* (more compatible with human interpretation) to represent sparse data through possibilistic representations. By doing so, an analyst acknowledges the vagueness and ambiguity inherent in the data. In conclusion, to quote the authors of the original meta-analysis study [Lubin and Boice, 1997],

> "Meta-analyses are valuable for identifying differences among studies and for summarizing results, but they should be interpreted cautiously when expected RRs are low as with indoor radon exposure, when there is study heterogeneity and where there is potential for confounding and exposure misclassification."

This further corroborates our proposition that possibilistic models offer viable alternatives to probabilistic models in analyzing meta-analysis data where information is limited and there is no suggested randomness between the studies. In possibilistic models data can be used as it is provided (intervals or point estimates) and hence do not require any intermediate assumptions to generate an uncertainty distribution.

4.1.2 Use of Possibility Distributions to Determine Exposure Concentrations in Epidemiological Studies

The broad purpose of individual epidemiological studies is to determine an

association between exposure and effects. The two types of epidemiological studies are ecologic and analytical case-control studies [National Academy Press, 1999], with the majority of the radon studies being case-control studies. A general methodology for case-control epidemiological studies consists of collecting data directly from the lung cancer cases and controls through personal interviews and monitoring of radon concentrations. Radon concentrations are monitored within each room/floor of the subjects' homes and their cumulative exposures over a period of 5-30 years prior to enrollment in the study are determined by extrapolating results of current exposures to past exposures. Association between exposure and effects are usually determined by estimating the increase in odds of lung cancer incidence in cases when compared to those in controls. The rate at which these odds change with categorical exposure levels is predicted using regression models such as log-linear or linear regression models [National Academy Press, 1999]. A number of factors such as the inability to accurately measure radon concentrations from current and past residences, imprecise knowledge of mobility of the subjects, and unknown effects of confounding factors such as smoking, education, gender and age, contribute significantly to the uncertainties in the final result of the odds ratio.

Rather than disputing the accuracy of any study, the purpose of our case study is to illustrate the application of possibility theory to represent uncertainties in exposure concentrations and to show the convergence of discrete possibility distributions to more continuous distributions with the availability of more data. Through this case study we also show that possibility distributions need not be limited to only sparse data sets but can also be applied to larger data sets.

Data from the Iowa Radon Lung Cancer Study (IRLCS) [Field *et al*., 2000, 2002] are used to illustrate the application of our methods. In the IRLCS study, lung cancer risks from smoking and radon exposure were studied in female residents who were diagnosed with lung cancer and the rates were compared to those female residents who were not diagnosed with cancer. The categorical exposure variables used in the study were based on "working level month" exposures accumulated over the period of 5-19 years prior to enrollment in the study. A time window of 5-19 years was chosen based on the assumption of a 5-year latency period for cancer in similar studies on miners and a 19-year limit was to ensure the availability of the subjects for the study and for follow-ups [Field *et al*., 2000]. Among the many types of analysis conducted as part of the study, one was to determine the total cumulative exposures of radon. These exposures were calculated from the accumulation of exposures from residential as well as other radon exposures and were calculated using Equation 4-2 as,

$$WLM_y = \frac{\lambda}{170 \times 100} \sum_l T_{ly} c_l \qquad \textbf{(4-2)}$$

where, λ is the assumed equilibrium ratio (assumed to be 0.5), T_{ly} is the total time spent at location l in year y and c_l is the radon concentration at the location in pCi/L. The IRLCS study was extensive and included radon concentration data from multiple locations on all floors and rooms of various residential types. In this case study, however, we restrict the calculations to the radon concentrations as measured on the first floor of a one-story building. Further, only radon concentrations from residential exposure are considered in our case study. Total time spent at home and radon

concentrations data were available as point estimates only and more than 900 concentration data points and more than 9000 mobility values were considered (result of data for each year of 15 years prior to enrollment).

To construct reasonable possibility distributions, cutoff points were determined with respect to the distance from the anchor. These cutoff points represent the cost functions determined by the data points that are most similar to each other. For example, cutoff points for the fraction of time spent at home were 1 through 9 in unit increments and represent the cost associated with combining the measurements. Each cutoff point forms an interval by including all values that are closest to the anchor, in other words, are most similar to the most possible value. At a cutoff point all values that are within a distance of 1 unit are combined together and subsequently all those that are at a distance of 2 units from the anchor are combined together in the next interval and so forth. The intervals derived from such agglomeration are shown in Table 4-7. By progressing through all the cutoff points one can derive all the consonant intervals. Weights are then assigned to each of the intervals by calculating the proportion of data points within each interval. Table 4-7 and Table 4-8 show the results obtained from this analysis, and Figure 4-6 and Figure 4-7 show the nested intervals along with the possibilistic weights. Once the possibility distributions for individual variables were determined, the exposures were then calculated using Equation 4-2. Exposures were calculated by using interval analysis techniques on intervals of time spent at home and concentrations at five possibilistic levels of 1, 0.7, 0.4, 0.2 and 0.07. For example, at the possibility level of 0.7,

$$\text{Exposure in WLM/yr} = \frac{0.5}{170 * 100} * [0.63, 0.80] * [1.45, 2.9] = [0.24, 0.60].$$

Other intervals at the corresponding possibilistic levels are calculated in a similar fashion.

The resulting possibility distribution is shown in Table 4-9 and Figure 4-8.

Table 4-7: Nested intervals derived for the fraction of time spent on the first floor of a one-story residential building in the IRLCS study

Intervals	Interval Weights	Possibilistic Weights
[0.67, 0.76]	0.21	1
[0.63, 0.8]	0.22	0.79
[0.59, 0.84]	0.21	0.57
[0.51, 0.92]	0.16	0.35
[0.47, 0.96]	0.12	0.2
[0.42, 1]	0.05	0.07
[0.38, 1]	0.01	0.02
[0.35, 1]	0.01	0.01
[0.08, 1]	0	0

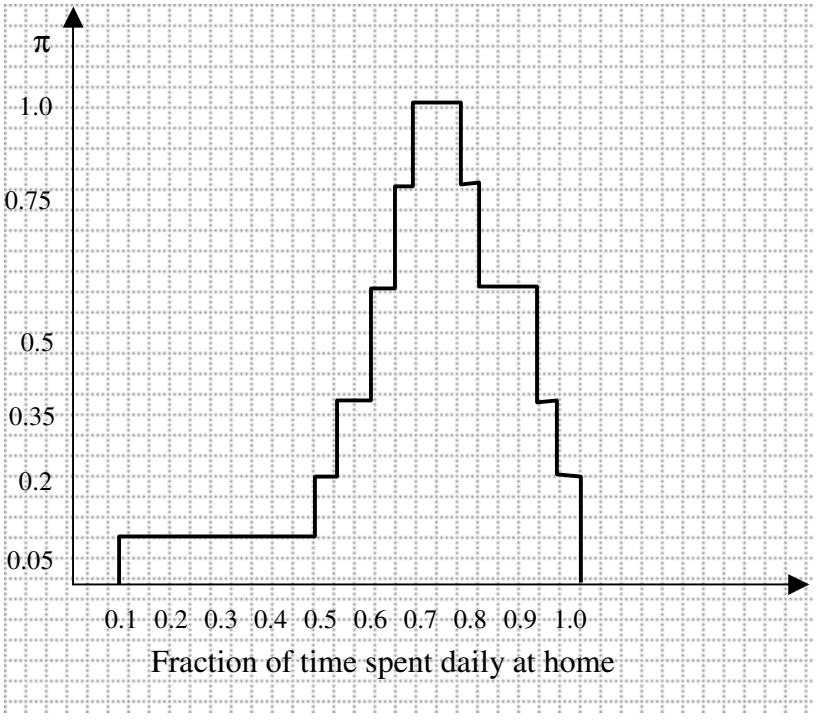


Figure 4-6: Possibility distribution for the fraction of time spent at on the first floor of one-story residential buildings in the IRLCS study.

Table 4-8: Nested intervals derived from the measurements of radon concentrations determined on the first floor of one-story residential buildings in IRLCS study

Intervals	Interval Weights	Possibilistic Weights
[1.95, 2.4]	0.13	1
[1.7, 2.65]	0.14	0.87
[1.45, 2.9]	0.18	0.73
[1.17, 3.15]	0.11	0.55
[0.95, 3.4]	0.14	0.44
[0.67, 3.65]	0.11	0.3
[0.43, 3.9]	0.13	0.19
[0.3, 4.15]	0.04	0.07
[0.3, 4.2]	0.03	0.03

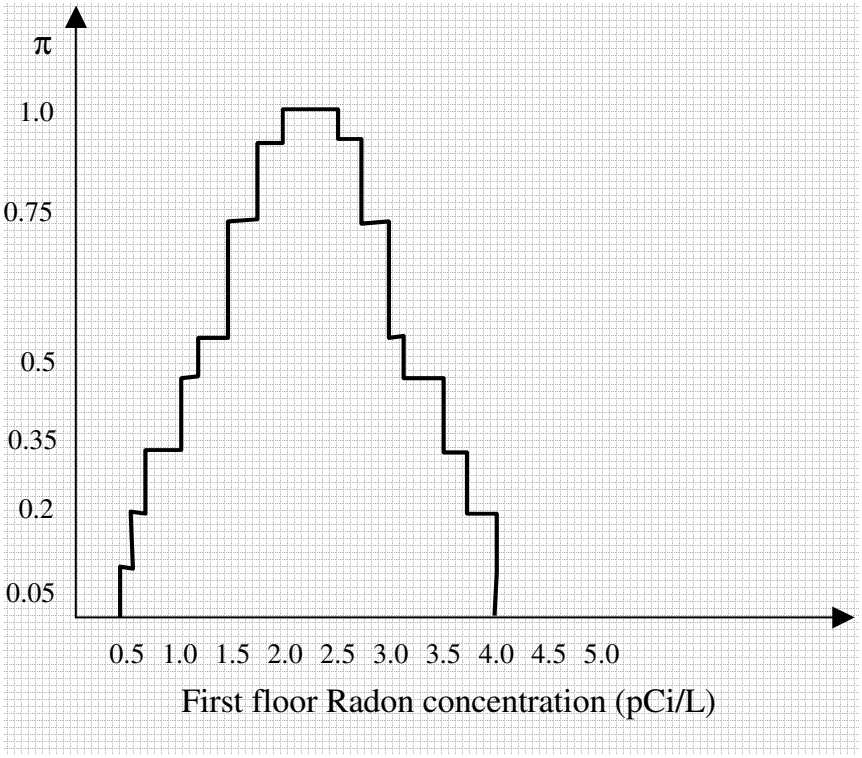


Figure 4-7: Possibility distribution for measurements of radon concentrations determined on the first floor of one-story residential buildings in IRLCS study.

Table 4-9: Residential cumulative exposures as determined from the possibility distributions of radon concentrations and mobility values from Equation 4-1

Exposure (WLM/yr)	Possibilistic Weights
[0.36, 0.47]	1
[0.24, 0.60]	0.7
[0.14, 0.74]	0.4
[0.08, 0.9]	0.2
[0.03, 1.07]	0.07

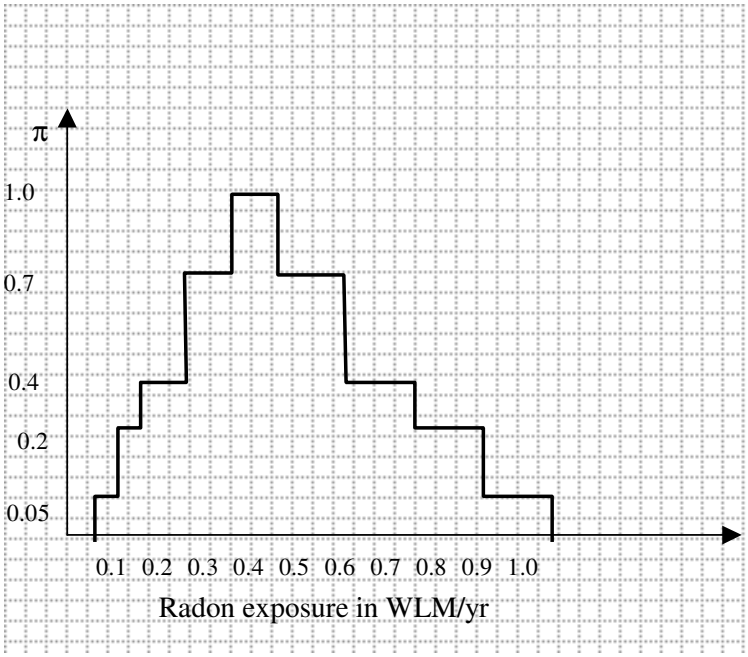


Figure 4-8: Possibility distribution of radon exposures in WLM/yr derived from individual possibility distributions of concentration and fraction of time spent on the first floor.

The possibility distribution generated by the combination of individual possibility distributions was determined by applying interval analysis methods on intervals at five different possibility levels (see Section 2.3.3 for more on interval methods).

4.2 Comparison of Results

To compare the possibilistic values to those obtained through probabilistic

analysis, we calculated the mean and standard deviation from running 1000 Monte Carlo simulations (MCS) of the concentrations and time spent data from section 4.1.2. Figure 4-9 shows the results of the analysis. A comparison of these values with those obtained by possibility methods indicates that the most possible value as expected is close to the mean obtained through MCS method (due to the calculation of the anchor as a statistical mean).

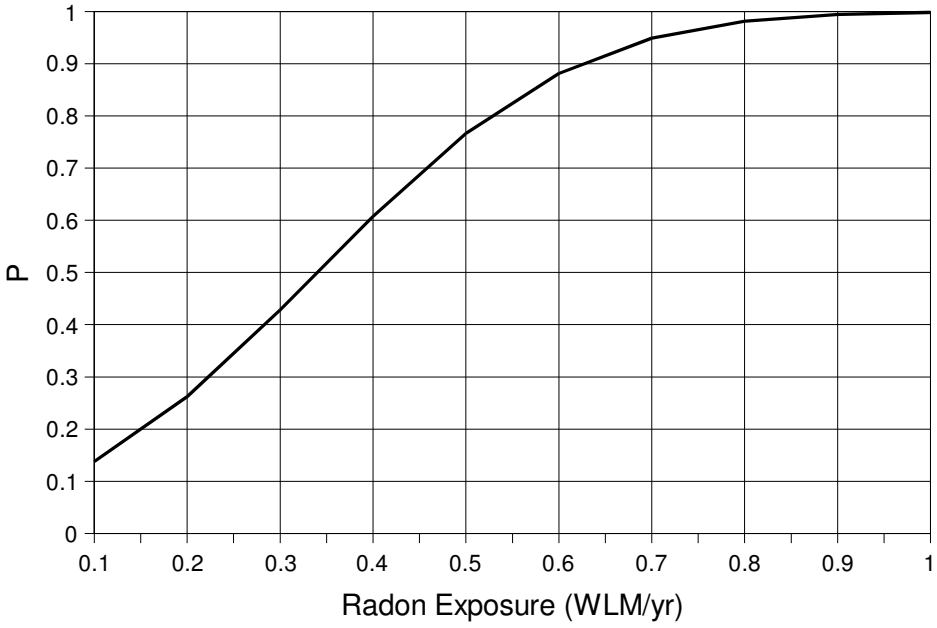


Figure 4-9: Radon exposures in WLM/yr as determined by Monte Carlo simulation with the fraction of time and concentrations expressed as probability distributions.

When considering the occurrence of any event, possibility yields estimates that are dependent on how similar the event is to the most possible event. For example, if one is to determine the probability of event "A" occurring, where "A" is the range of exposures between 0.2 and 0.3 WLM/yr, in probability theory this is calculated from the cumulative distribution function in Figure 4-9, as $P(0.2 \leq X \leq 0.3) = 0.43 - 0.26 = 0.17$.

In contrast, using the *possibility* distribution, the values are calculated from Figure 4-8 as, $\Pi(0.2,0.3) = 0.7$. The probability results indicate that the answer is not dependent on the most likely value, which in most cases is the mean value; on the other hand, the possibility results show that as one approaches the most likely value the possibility of it occurring increases. This seems like a more intuitive answer since as one approaches the value that is most likely to occur their confidence in the range should increase, and this confidence would correspondingly decrease as they move away from the most likely value. From Figure 4-8 it can also be seen that the intervals at various possibility values denote the certainty with which we are sure that the most possible value lies in that range. For example, the certainty that the value will be in the interval [0.12, 0.6] is calculated from Equation 3-8 as, $N(A) = 1 - \Pi(\overline{A}) = 0.6$. Therefore, given the data, an expert is 60% certain that the most possible value will be in the interval [0.12, 0.6].

Thus using a possibilistic approach one automatically can deduce the bounds on the intervals at various levels of certainty. In regular risk assessments or risk management studies, these values can then be propagated with ease using methods such as interval methods and aid the analyst in making decisions in the midst of ambiguity

Chapter 5 Conclusions

Probabilistic techniques, either subjective or objective, handle uncertainty in system behavior by assuming that the evidence revealed by the system is specific and accurate enough to merit the generation of additive probability measures. When however, the assumption of the specificity with which a system functions is based on evidence that really is imprecise, ambiguous or incomplete, then assuming a probabilistic model can lead to accepting values that are not necessarily true representations of the system. In systems such as risk assessment, analysis of the sources of uncertainty reveals that randomness is only one type of uncertainty, and a more general uncertainty model is required to include all the types. By relaxing the axiom of additivity, possibility theory offers a promising and general model to acknowledge and include ambiguity, ignorance and imprecision of evidence in uncertainty distributions. The usefulness of possibility theory is especially crucial in areas where evidence is very limited and the measurements reveal information that is ambiguous and vague. Measurements acquired about a variable of interest are usually not specific values and even if so, the measurements seldom represent complete evidence and the underlying uncertainty distribution is often unknown. The vagueness or imprecision in the measurements can

result either from the characteristics of the measurements themselves or in what they infer. In other words, when the measurements are intervals, as when the instrument (objective or subjective) is imprecise, the uncertainty arises due to the instrument imprecision itself, however, when the instrument is capable of producing precise measurements but the content of the measurements reveals ambiguity, the uncertainty is due to the data itself. In either case, the distribution needs to quantitatively reflect the amount of imprecision and vagueness and possibility distributions offer a more natural representation of uncertainty.

In this dissertation, we specifically focused on laying foundations for deriving possibility distributions from three types of imprecise empirical evidence structures:

1. When evidence is represented by consistent measurements or non-consistent and non-disjoint measurements in the form of overlapping intervals.
2. When evidence is in the form of sparse and unique point estimates.
3. When evidence is available in the form of disjoint interval measurements.

Two novel methodologies were developed to construct possibility distributions from these three types of measurements. Due to the underlying difference in the structural form of each of these measurement types, separate methods had to be developed to derive the possibilistic histogram. While consistent measurements are assumed to reflect evidence that is ambiguous (occurrence of the same value in two different sets), point estimates (also called singletons) and disjoint interval measurements are reflective of evidence that is conflicting (occurrence of two completely different values such as is the case for probabilistic models). For consistent or overlapping measurements, evidence tends to confirm a certain set of recurring values; therefore possibility distributions were

derived through a new method, called Method I, based on the intersection of sets. In contrast, in our Method II, since the measurements that are disjoint state the complete conflict in evidence, possibility distributions were derived using hierarchical clustering techniques based on similarity concepts and a distance metric. Method II, which groups values similar to each other through the application of a cost function, was used along with possibilistic principles to derive intervals such that the possibilistic weights of each interval reflected the quantity of observed empirical data that actually supports this interval. Through this process a discrete graded contour function that revealed the true uncertainty in the system was generated over the resulting consonant intervals.

It is also possible to use these two methods in combination when measurements are more general, i.e., they are not specifically of any one type outlined in the previous paragraph. As shown in Figure 5-1, the measurements can be a collection of disjoint intervals, singletons, and non-disjoint and/or non-consistent intervals. Such a collection can arise when measurements are derived from various sources and a possibility distribution needs to be constructed from these measurements. In such a case, depending on the pattern of measurements, the two methods developed in this dissertation can be used separately without any modifications or they can be combined to provide a hybrid method where Method I can be used to generate a set of all the possible intersections and Method II can be used to cluster the intervals and generate a possibility distribution.

An issue to note in this hybrid approach, however, is that Method I usually results in a more constrained possibility distribution biased over all the consonant intervals while Method II results in a wider possibility distribution. Method I yields a narrower and more tapered distribution due to the fact that the consonant intervals dominate the non-

consonant intervals in the set of all intersections, and the weight is transferred from the non-consonant to the consonant portions. This might be a desirable characteristic when one intends to let the evidence lead to the most possible value by assigning more weight to the consonant portion of the data and less weight to the non-consonant portion. Therefore, the data that supports and confirms other data sets is given more weight than that portion which is conflicting. In contrast, Method II results in a wider distribution, and by virtue of the agglomerative nature of clustering can result in combining intervals that are consonant into one single interval. This can result in losing the consonance of the intervals and can result in a distribution that tapers more gradually to the most possible. Method II, however, results in a more uniform distribution of weight, and is more useful when consonance is assumed to be from the entire data set and no corroboration of evidence between measurements is assumed. The choice of the methods is therefore dependent on the application, and the analyst should choose the appropriate method based on the intent of the analysis, measurement characteristics, and data content.

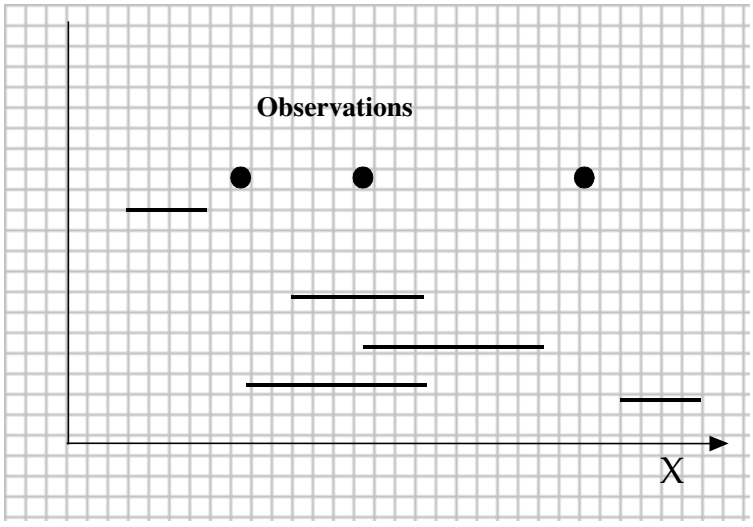


Figure 5-1: An example of a general collection of measurements.

The utility and the importance of the two new methods developed herein were illustrated through a meta-analysis case study in human health risk assessment that was conducted by pooling 8 individual studies to determine relative cancer risk values from residential exposure to radon gas. It was shown through the comparison with results from the original meta-analysis study that possibility distributions are more natural representations of sparse imprecise information than are probability distributions. In particular, possibility distributions allow an analyst to depict the state of the uncertainty more realistically by suggesting two measures, possibility and necessity. It was shown that one could quantitatively interpret ambiguous and imprecise data, and portray what is known about the uncertainty through possibility distributions and then use these two new methods to find the bounds on the true value.

Probability theory provides confidence limits to bound the true value. In contrast, possibility theory offers possibility and necessity measures to bound the true value. In comparison to probability theory, where the confidence limits denote the confidence in the range within which the true value can occur, the necessity value in possibility theory indicates the certainty with which an analyst can indicate that the true value lies in the interval. Hence, probability theory assumes a probability distribution to characterize the random behavior in the system, whereas possibility theory assumes that an analyst is ambiguous about what the true outcome could be and therefore derives a distribution based on ambiguity and vagueness in the data. Based on this distinction, the first case study on radon gas exposure (Section 4.1.1.1) showed that using possibilistic analysis without any assumptions, one is only 25% certain (necessity is 0.25) in the interval that was derived using probability theory as the 95% confidence limit in the original study.

Another way of construing this necessity measure is that only 25% of the available evidence completely supports the occurrence of the true value being within the 95% confidence interval of the original study. This indicates that using the available evidence, rather than requiring one to assume and infer from a random distribution, can yield much lower confidence in the results.

To illustrate another distinction, in a possibility distribution the possibility of a particular value increases as it approaches the most possible value, with the possibility reaching 1.0 when the value lies in the most possible interval. The cumulative probability distribution function (CDF) superimposed over a possibility distribution (CDF shown as the dotted line in Figure 5-2) illustrates this difference between a possibility and probability distribution that were derived from the same empirical data set. For example, if the most possible value (within the interval where $\pi=1$) is representative of the mean of the data, then the possibilities of values close to the mean should be higher than those further from the mean. This interpretation of the mean of the data, that which lies within the most possible interval, needs to be distinguished from the mean that is derived in the probabilistic sense. In the latter case, the mean value occurs in 50% of the cases; in possibility the most possible interval simply asserts that there is a complete possibility of the value occurring. Therefore, the mean value in the possibilistic model is an assertion that the most possible value lies in an interval that is closest to the mean and no other precision is assumed. This lack of precision is not a reflection of inaccuracy but merely a reflection of the realities of the data.

Now, from the CDF in Figure 5-2, the probability that the value would lie in the

range [0.2,0.3] is, $P(0.2 \le X \le 0.3) = 0.43 - 0.26 = 0.17$, while the possibility is $\Pi([0.2,0.3]) = 0.7$. Thus, the possibility of a value being the actual radon exposure increases as one tends towards the most possible value. This higher value of possibility seems to be consistent with natural thought, in that, as one approaches the most likely value the possibility of that value should increase.

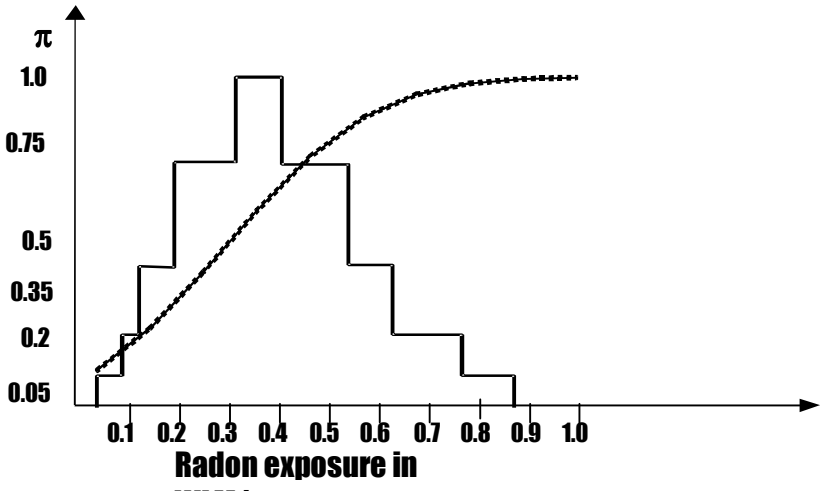


Figure 5-2: Comparison of possibilistic and probabilistic results derived from the data set as given in Section 4.1.2. The ordinate represents both possibility as well as cumulative probability. The mean value from the CDF is 0.34, while the most possible value from the possibility distribution is in the interval [0.3, 0.38].

Therefore, by relying on what is revealed by the evidence, possibility distributions offer a more natural representation of uncertainty and reflect this uncertainty realistically through non-additive measures. Due to their independence from the stricter law of additivity, possibility distributions can be derived from a wide range of measurement types and can be conveniently used to model uncertainty even when data is very sparsely available. As shown in Section 4.1.2 of Chapter 4, possibility distributions can also be derived from large data sets where the ambiguity and vagueness result from interpretation

of the data and not from the measurements themselves.

Future Research

Though the research presented in this dissertation addresses one of the critical limitations of possibility theory, that of deriving possibility distributions from empirical data, there is more development that could be done. Due to the flexibility of possibility theory to model the uncertainties from various kinds of data and its usefulness in complex systems, it is appropriate to further invest in its advancement so it can be more useful in practice rather than just being a theoretical exercise. A list of the areas where the advancement can be beneficial to the scientific community is presented below. This list is definitely not meant to be exhaustive, but only intended to identify some areas to initiate more research.

- One of the promising areas of application of possibility theory is in regression analysis. Regression techniques are often based on imprecise and ambiguous data and traditional methods of deriving point estimates for regression coefficients falls short of truly representing the uncertainty in the model. Uncertainty in the traditional models is sometimes represented by assuming a distribution or through standard error terms that only capture randomness in the data. And current fuzzy regression techniques [Tanaka *et al.*, 1982; Ross, 1995] derive regression coefficients as fuzzy sets or as possibility distributions that are based on parametric distributions and are derived using complex linear programming

methods. However, there is scope for further enhancements of these models to generate more generic and intuitive distributions.

- Issues such as correlation between possibilistic variables, conditional dependency, possibilistic updating and combination need to be explored further in the context of empirical possibility distributions. Currently, several techniques exist for those distributions that have more subjective interpretations [Sandri *et al*., 1995], however similar techniques need to be developed and validated for empirical distributions.
- Techniques such as neural networks and genetic algorithms could offer other alternatives to developing nested interval structures. For example, a neural network could be trained to extract possibilistic intervals and weights from empirical data.
- There has been at least one research project on the use of uncertainty measures to facilitate the derivation of possibility distributions from Method I presented in this dissertation [Chavez, 2002], however, this effort was limited to consistent measurement types. Further computational research that incorporates all the measurement types and other methods in this dissertation can assist in identifying the most optimal and appropriate possibility distributions.
- In some models, a combination of distributions might be available, i.e., probabilistic distributions might be available on some variables while on some others possibilistic distributions might be known. In such cases, it might be beneficial to develop an approach to propagate this combined information within

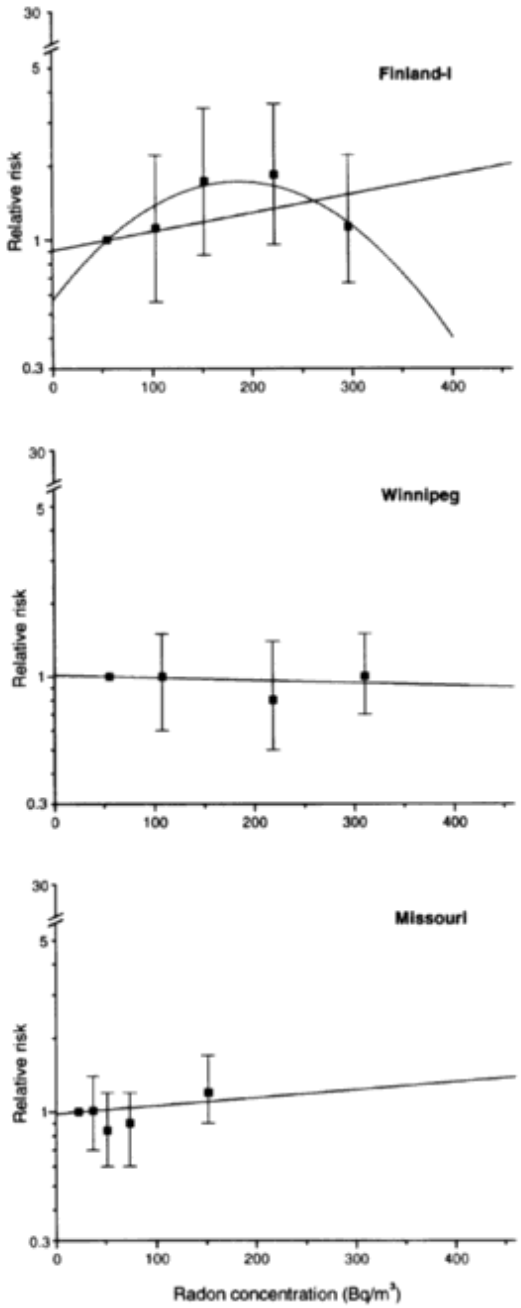
a system. One idea might be to use Monte Carlo techniques with interval methods to propagate such combinations.

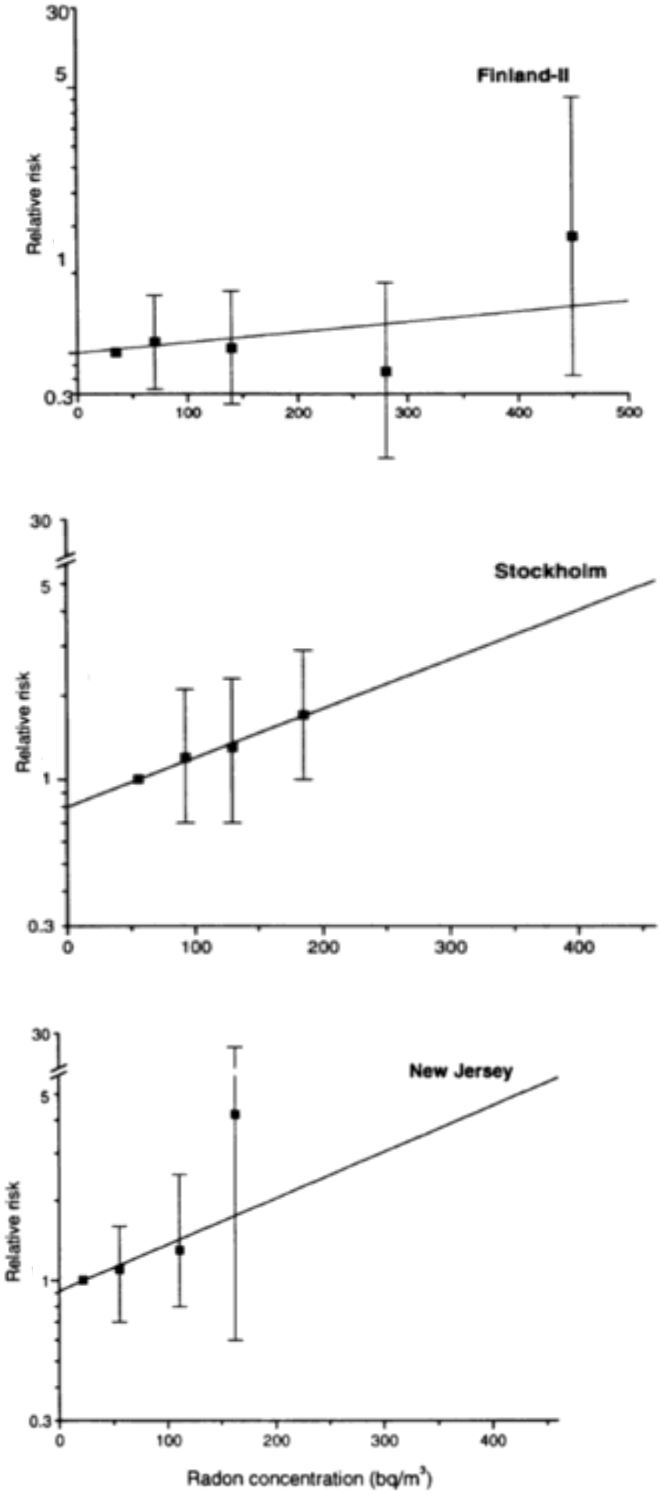
- Finally, the methods developed in this dissertation were restricted to theoretical developments and therefore an apparent area of future work is the development of efficient computational methods to implement the algorithms.

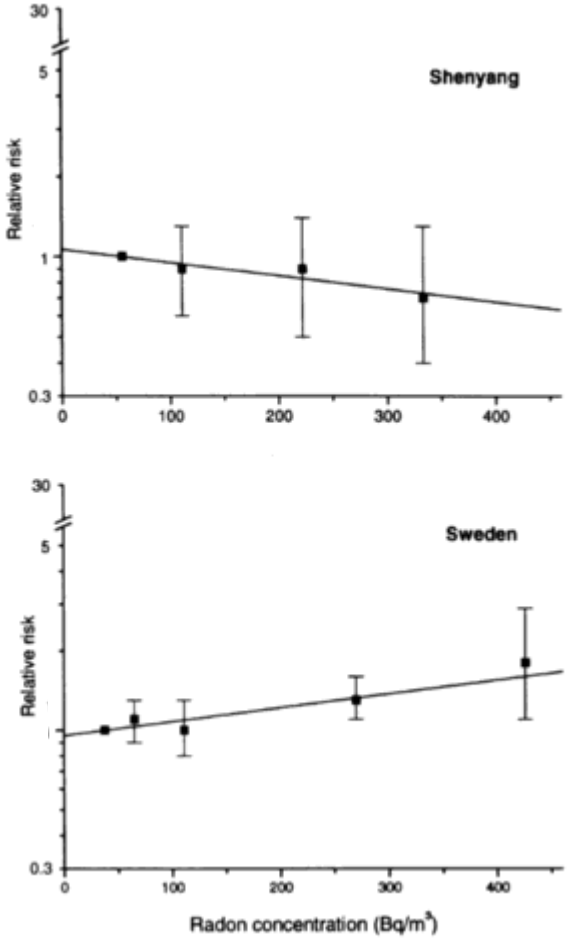
APPENDICES

APPENDIX A - Summary of epidemiological studies used in the case study

Figure A - 1: Relative risk values and the fitted exposure response models for each of the 8 residential epidemiological studies [Lubin and Boice, 1997]. The data from these curves was used in case study outlined in Section 4.1.1.




Finland-II
Stockholm
New Jersey
Relative risk
30
5
1
0.3
0
100
200
300
400
500
Radon concentration (bq/m³)


Shenyang
Relative risk
30
5
1
0.3
0
100
200
300
400
Sweden
Relative risk
30
5
1
0.3
0
100
200
300
400
Radon concentration (Bq/m³)

APPENDIX B - Mathematical Notation

i, j, k: Arbitrary indices

m : Mass on random sets or random intervals

p: Probability mass function

w : weights on observed measurements

x : Singletons on a real line

x_c : Anchor for singleton measurements

x^L, x^U : End points of a random interval defined on a real line $\Re$

A : Random interval or an event

$\tilde{A}$: Fuzzy set

A_λ : λ-cut interval on a fuzzy membership function

B: Random intervals derived from conjunction of observed measurements

Bel: Belief measure

C: Consonant intervals derived from disjoint measurements

F : Focal set

G: A set formed by a collection of B

H: Consonant set derived from intersections of original measurements

I: Non-consonant set derived from intersections of original measurements

K: 1-K is the normalization factor in Dempster-Shafer theory

M: Number of observed measurements

N: Number of random intervals for which $m > 0$

O: Disjoint measurement intervals

O_{New}: New interval formed by aggregating original disjoint measurements

P: Probability measure

Pl: Plausibility measure

Q: Total number of intersecting sets derived from original measurements

Q_H: Number of consonant sets in Q

Q_I: Number of non-consonant sets in Q

S: Support of a possibility distribution

S_R: Random set

X: Universe of discourse on the real line

X^a: Anchor for disjoint interval measurements

$\Im$: σ-algebra

$\wp(\Omega)$: Power set on Ω

$\Re$: Set of real numbers

β: Similarity function in the context of possibility distributions

δ: Distance metric

η: Normalized ν

κ: Redistribution factor

μ: Fuzzy membership function

ν: Weights on B

π: Possibility distribution

ρ: Redistribution weight

υ : Fuzzy measure

ω : Elements of Ω

Δ : Cost function

N : Necessity measure

Π : Possibility measure

Ω : Universe of discourse

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